II. *The Potential of an Anchor Ring.*

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Introduction.

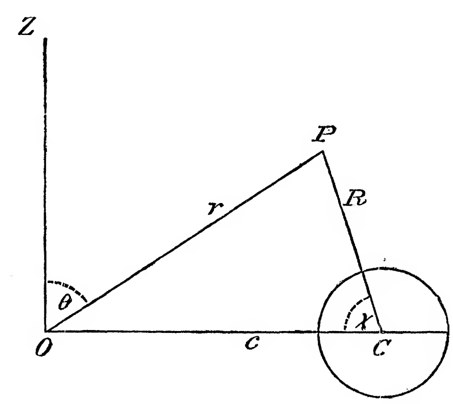
IN this Paper I have developed a method of dealing with questions connected with Anchor Rings.

If r, θ, ϕ be the coordinates of any point outside an anchor ring, whose central circle is of radius c, then

$$\int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\sin\theta\cos\phi)}}$$

is a solution of LAPLACE's equation, finite at all external points and vanishing at infinity. Let this be called I. Then $d\mathrm{I}/dz$ is another solution; and two sets of solutions may be found by differentiating I and $d\mathrm{I}/dz$ any number of times with respect to c. These solutions are symmetrical with respect to the axis of the ring. In the first set z is involved only in even powers; in the second set in odd powers.

Take a section through the axis Oz of the ring and the point P, (r, θ) cutting the central circle of the ring in C.



Let CP = R and $\angle$ OCP = χ.

When R is less than c, the integral

$$\int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\sin\theta\cos\phi)}}$$

is expanded in ascending powers of R/c, and the expansions of the integrals, obtained by differentiating this with respect to c, are deduced. Section I. is devoted to the discussion of these functions, and some similar ones, needed in hydrodynamic applications.

In Section II. the potential of an anchor ring at all external points is found in a very convergent series of integrals. The expansions of Section I. are not needed; but the first few terms are reduced to elliptic integrals. The equipotential surfaces are drawn for the ratios $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, 1 of the thickness of the ring to its mean diameter.

In Section III. the potential of a conductor, in the form of an anchor ring, is found at external points; the surface density at any point of the ring and the charge are also determined. Section IV. consists of a discussion of the motion of an anchor ring in an infinite fluid; the velocity potential or stream line function for motion parallel to the axis, perpendicular to the axis, &c., being first determined. The kinetic energy is determined in the several cases; and in the case of the cyclic motion through the ring, the linear momentum. In this last case, the solid angle subtended by a circle at a point near a circumference, is incidentally found.

In Section V. the annular form of rotating fluid is discussed, when the thickness of the annulus is small compared with its mean radius.

It is shown that the form of the cross section may be represented by $R = a(1 + \beta_2 \cos 2\chi + \beta_3 \cos 3\chi +, \&c.)$, where β_2, β_3, &c., are of the second, third, &c., order in a/c. Their values are found as far as $(a/c)^4$.

To the second order

$$\frac{\omega^2}{\pi\rho} = \left(\frac{a}{c}\right)^2 \left(\log \frac{8c}{a} - \frac{5}{4}\right),$$

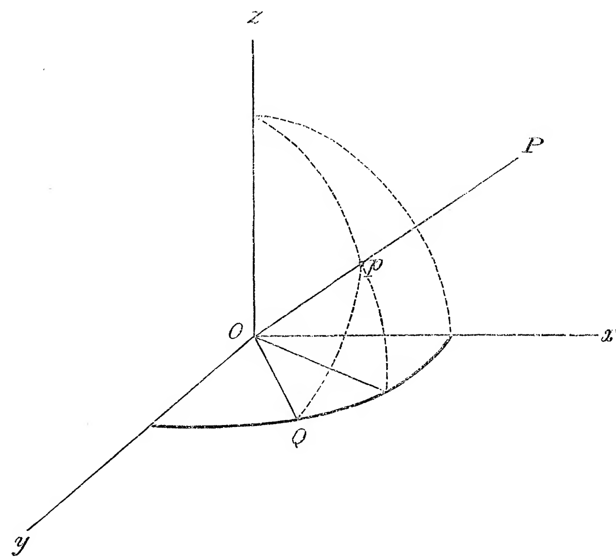
$$\beta_2 = \frac{5}{8}\left(\frac{a}{c}\right)^2 \left(\log \frac{8c}{a} - \frac{17}{12}\right).$$

The method employed throughout the paper has not the analytical elegance of Mr. HICKS' Toroidal Functions, but it has many advantages. The potential of an attracting ring takes a very simple form. The boundary conditions to be satisfied in hydrodynamical applications are very simple. The results are obtained in terms of R and χ, the quantities most obviously connected with a ring.

I am greatly indebted to Mr. HERMAN, Fellow and Assistant Tutor of Trinity College, Cambridge, for the careful manner in which he has read over much of the work, and the many errors he has corrected. In consequence, the paper will, I think, be found free from any serious mistakes.

Section I.—*Preliminary Analysis.*

§ 1. Take a fine ring in the plane of x, y, centre at the origin, and consisting of attracting matter of density $k \cos n\phi'$, where ϕ' is the azimuth of any point.



The potential of this ring at a point P, whose coordinates are r, θ, ϕ, is

$$\int_0^{2\pi} \frac{k \cos n\phi' \, c d\phi'}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos(\phi - \phi')\}}}.$$

Put $\phi' = \phi + \psi$, then the potential

$$= \int_0^{2\pi} \frac{ck (\cos n\phi \cos n\psi - \sin n\phi \sin n\psi) \, d\psi}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos\psi\}}}$$

$$= \cos n\phi \int_0^{2\pi} \frac{ck \cos n\psi \, d\psi}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos\psi\}}}$$

as the second integral vanishes between the limits.

Therefore

$$\cos n\phi \int_0^{\pi} \frac{\cos n\phi \, d\phi}{\sqrt{(r^2 + c^2 - 2cr \sin\theta \cos\phi)}}$$

or

$$\cos n\phi \int_0^{\pi} \frac{\cos n\phi \, d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos\phi + \varpi^2)}}$$

is a solution of LAPLACE's equation.

Similarly

$$\sin n\phi \int_0^\pi \frac{\cos n\phi \, d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos\phi + \varpi^2)}}$$

is a solution.

Let V stand for either of these integrals. Then

$$\frac{d^{p+q}\,\mathrm{V}}{dc^p \, dz^q}$$

is also a solution of LAPLACE's equation.

For different values of p and q the solutions are not all independent: as it is easily seen that

$$\int_0^\pi \frac{\cos n\phi \, d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos\phi + \varpi^2)}}$$

satisfies a linear partial differential equation of the second order in c and z.

But two independent sets of solutions are obtained by giving different values to p in $\frac{d^p\mathrm{V}}{dc^p}$ and $\left(\frac{d}{dc}\right)^p \cdot \left(\frac{d\mathrm{V}}{dz}\right)$.

The only cases considered in this paper are when $n = 0$ or $n = 1$. That is the solutions of the forms

$$\left(\frac{d}{dc}\right)^p \int_0^\pi \frac{d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos\phi + \varpi^2)}};$$

$$\left(\frac{d}{dc}\right)^p \frac{d}{dz} \int_0^\pi \frac{d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos\phi + \varpi^2)}};$$

$$\cos\phi \left(\frac{d}{dc}\right)^p \int_0^\pi \frac{\cos\phi \, d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos\phi + c^2)}};$$

and

$$\cos\phi \left(\frac{d}{dc}\right)^p \frac{d}{dz} \int_0^\pi \frac{\cos\phi \, d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos\phi + \varpi^2)}};$$

as these cover all the cases to which the functions are applied in this paper.

It is easily seen that

$$\varpi \int_0^\pi \frac{\cos\phi \, d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos\phi + \varpi^2)}}$$

and, consequently,

$$\varpi \left(\frac{d}{dc}\right)^p \int_0^\pi \frac{\cos\phi \, d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos\phi + \varpi^2)}}$$

are solutions of the equation

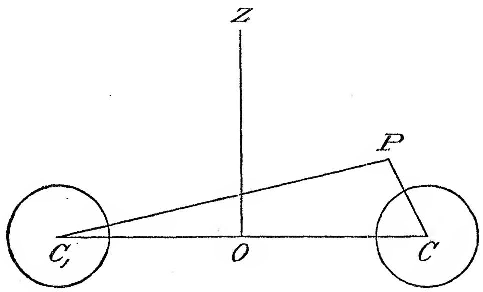
$$\frac{d^2\psi}{dz^2} + \frac{d^2\psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\psi}{d\varpi} = 0.$$

For putting $\psi = \varpi\phi$, this equation becomes

$$\frac{d^2\phi}{dz^2} + \frac{d^2\phi}{d\varpi^2} + \frac{1}{\varpi}\frac{d\Phi}{d\varpi} - \frac{1}{\varpi^2}\Phi = 0,$$

that is, $\Phi \cos\phi$ is a solution of LAPLACE'S equation.

§ 2. *Expansion of the Functions.*—Let the plane through the axis of the ring and a point P, cut the ring in the two circles whose centres are C and C_1.



Let

$$CP = R\,; \quad C_1P = R_1\,; \quad OC = c\,;$$

and the angle

$$OCP = \chi.$$

Also, for convenience, put

$$s = \frac{R}{c},$$

and

$$l = \log\frac{8c}{R} - 2.$$

When R is less than c the above integrals may be expanded in ascending powers of R/c or s.

$$\int_0^\pi \frac{d\phi}{\sqrt{\{r^2 + c^2 - 2cr\sin\theta\cos\phi\}}}$$

$$= \frac{2}{R_1}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\left\{1 - \frac{R_1^2 - R^2}{R_1^2}\sin^2\phi\right\}}}$$

$$= \frac{2}{R_1}\left\{\log\frac{4R_1}{R} + \frac{1^2}{2^2}\frac{R^2}{R_1^2}\left(\log\frac{4R_1}{R} - \frac{2}{1.2}\right) + \frac{1^2 3^2}{2^2 4^2}\frac{R^4}{R_1^4}\left(\log\frac{4R_1}{R} - \frac{2}{1.2} - \frac{2}{3.4}\right) + \ldots\right\}.$$

[CAYLEY, 'Ell. Func.,' p. 54.]

$$\int_0^\pi \frac{\cos\phi\, d\phi}{\sqrt{\{r^2 + c^2 - 2cr\sin\theta\cos\phi\}}}$$

$$= \frac{2}{R_1}\int_0^{\frac{\pi}{2}} \frac{(2\cos^2\phi - 1)\,d\phi}{\sqrt{\left\{1 - \frac{R_1^2 - R^2}{R_1^2}\cos^2\phi\right\}}}$$

$$= \frac{2}{R_1}\int_0^{\frac{\pi}{2}} \frac{2\left(\cos^2\phi - \frac{R_1^2}{R_1^2 - R^2}\right) - 1 + \frac{2R_1^2}{R_1^2 - R^2}}{\sqrt{\left\{1 - \frac{R_1^2 - R^2}{R_1^2}\cos^2\phi\right\}}}\,d\phi,$$

$$= \frac{2}{R_1}\left\{\frac{2}{1 - \frac{R^2}{R_1^2}}\left[\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\left\{1 - \frac{R_1^2 - R^2}{R_1^2}\sin^2\phi\right\}}} - \int_0^{\frac{\pi}{2}} \sqrt{\left\{1 - \frac{R_1^2 - R^2}{R_1^2}\sin^2\phi\right\}}d\phi\right] - \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\left\{1 - \frac{R_1^2 - R^2}{R_1^2}\sin^2\phi\right\}}}\right\}$$

$$= \frac{4}{R_1\left(1 - \frac{R^2}{R_1^2}\right)}\left[\left\{\log\frac{4R_1}{R} + \frac{1^2}{2^2}\frac{R^2}{R_1^2}\left(\log\frac{4R_1}{R} - \frac{2}{1.2}\right) + \dots\right\}\right.$$

$$\left. - \left\{1 + \frac{1}{2}\frac{R^2}{R_1^2}\left(\log\frac{4R_1}{R} - \frac{1}{1.2}\right) + \frac{1^2.3}{2^2.4}\frac{R^4}{R_1^4}\left(\log\frac{4R_1}{R} - \frac{2}{1.2} - \frac{1}{3.4}\right) + \dots\right\}\right]$$

$$- \frac{2}{R'}\left\{\log\frac{4R_1}{R} + \frac{1^2}{2^2}\frac{R^2}{R_1^2}\left(\log\frac{4R_1}{R} - \frac{2}{1.2}\right) + \frac{1^2.3^2}{2^2.4^2}\frac{R^4}{R_1^4}\left(\log\frac{4R_1}{R} - \frac{2}{1.2} - \frac{2}{3.4}\right) + \dots\right\}$$

[CAYLEY, 'Ell. Func.,' p. 54.]

$$= \frac{2}{R_1}\left\{\log\frac{4R_1}{R} - 2 + \frac{R^2}{R_1^2}\frac{5\log\frac{4R_1}{R} - 7}{4} + \frac{R^4}{R_1^4}\frac{162\log\frac{4R_1}{R} - 225}{128} + \&c.\right\}.$$

Now

$$R_1^2 = 4c^2 - 4cR\cos\chi + R^2$$
$$= 4c^2\left(1 - \frac{R}{c}\cos\chi + \frac{R^2}{4c^2}\right)$$
$$= 4c^2\left(1 - s\cos\chi + \frac{s^2}{4}\right).$$

Therefore

$$\log\frac{4R_1}{R} = \log\frac{8}{s} - \frac{s}{2}\cos\chi - \frac{s^2}{4}\frac{\cos 2\chi}{2} - \frac{s^3}{8}\frac{\cos 3\chi}{3} - \&c.$$
$$= l + 2 - \frac{s}{2}\cos\chi - \frac{s^2}{8}\cos 2\chi - \frac{s^3}{24}\cos 3\chi - \&c.$$

$$\frac{R^2}{4R_1^2} = \frac{s^2}{16} + \frac{s^3}{16}\cos\chi + \frac{s^4}{64}(1 + 2\cos 2\chi) + \&c.$$

Substituting in the above expansions

$$\int_0^\pi \frac{d\phi}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos\phi\}}}$$

$$= \frac{2}{\mathrm{R}_1}\left\{l + 2 - s\frac{\cos\chi}{2} + s^2\left(\frac{l+1}{16} - \frac{\cos 2\chi}{8}\right) + s^3\left(\frac{2l+1}{32}\cos\chi - \frac{\cos 3\chi}{24}\right)\right.$$

$$\left. + s^4\left(\frac{50l+15}{2048} + \frac{4l+1}{128}\cos 2\chi - \tfrac{1}{64}\cos 4\chi\right) + \ldots\right.$$

$$\int_0^\pi \frac{\cos\phi\, d\phi}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos\phi\}}}$$

$$= \frac{2}{\mathrm{R}_1}\left\{l - s\frac{\cos\chi}{2} + s^2\left(\frac{5l+3}{16} - \frac{\cos 2\chi}{8}\right) + s^3\left(\frac{10l+1}{32}\cos\chi - \tfrac{1}{24}\cos 3\chi\right)\right.$$

$$\left. + s^4\left(\frac{322l+35}{2048} + \frac{20l-3}{128}\cos 2\chi - \tfrac{1}{64}\cos\chi\right) + \ldots\right\}.$$

Now

$$\frac{2}{\mathrm{R}_1} = \frac{1}{c}\left(1 - s\cos\chi + \frac{s^2}{4}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{c}\left\{1 + \frac{s\cos\chi}{2} + s^2\left(\frac{1}{16} + \frac{3}{16}\cos 2\chi\right) + s^3\left(\frac{3}{64}\cos\chi + \frac{5}{64}\cos 3\chi\right)\right.$$

$$\left. + s^4\left(\frac{9}{1024} + \frac{20}{1024}\cos 2\chi + \frac{35}{1024}\cos 4\chi\right) + \ldots\right\}.$$

Therefore

$$\int_0^\pi \frac{d\phi}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos\phi\}}}$$

$$= \frac{1}{c}\left\{l + 2 + s\frac{l+1}{2}\cos\chi + s^2\left(\frac{2l+1}{16} + \frac{3l+2}{16}\cos 2\chi\right)\right.$$

$$+ s^3\left(\frac{9l+3}{64}\cos\chi + \frac{15l+7}{192}\cos 3\chi\right)$$

$$\left. + s^4\left(\frac{105l+27}{2048} + \frac{60l+13}{768}\cos 2\chi + \frac{105l+34}{3072}\cos 4\chi\right) + \text{\&c.}\right\}.$$

And

$$\int_0^\pi \frac{\cos\phi d\phi}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos\phi\}}}$$

$$= \frac{1}{c}\left\{l + s\frac{l-1}{2}\cos\chi + s^2\left(\frac{6l+1}{16} + \frac{3l-4}{16}\cos 2\chi\right)\right.$$

$$+ s^3\left(\frac{33l+1}{64} + \frac{15l-23}{192}\cos 2\chi\right)$$

$$\left. + s^4\left(\frac{540l+27}{2048} + \frac{240l-23}{768}\cos 2\chi + \frac{105l-176}{3072}\cos 4\chi\right) + \text{\&c.}\right\}.$$

§ 3. To obtain the expansion of $\frac{d}{dc}\int_0^\pi \frac{d\phi}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos\phi\}}}$ in ascending powers of $\frac{R}{c}$, the corresponding expansion of $\int_0^\pi \frac{d\phi}{\sqrt{\{r^2 + c^2 - 2cr\sin\theta\cos\phi\}}}$ must be differentiated with respect to c, R and χ being considered as functions of r, θ, and c.

Now

$$\left.\begin{aligned} R\cos\chi &= c - r\sin\theta = c - \varpi \\ R\sin\chi &= r\cos\theta = z \end{aligned}\right\}.$$

Therefore

$$\left.\begin{aligned} \cos\chi\frac{\partial R}{\partial c} - R\sin\chi\frac{\partial\chi}{\partial c} &= 1 \\ \sin\chi\frac{\partial R}{\partial c} + R\cos\chi\frac{\partial\chi}{\partial c} &= 0 \end{aligned}\right\}.$$

Solving these equations

$$\frac{\partial R}{\partial c} = \cos\chi$$

$$R\frac{\partial\chi}{\partial c} = -\sin\chi.$$

Again

$$\cos\chi\frac{\partial R}{\partial z} - R\sin\chi\frac{\partial\chi}{\partial z} = 0$$

and

$$\sin\chi\frac{\partial R}{\partial z} + R\cos\chi\frac{\partial\chi}{\partial z} = 1.$$

Solving these equations

$$\frac{\partial R}{\partial z} = \sin\chi\ ; \quad R\frac{\partial\chi}{\partial z} = \cos\chi.$$

From these formulæ it follows that

$$\frac{d}{dc}(R^n\cos n\chi) = nR^{n-1}\cos(n-1)\chi$$

and

$$\frac{d}{dc}\left(\frac{\cos n\chi}{R^n}\right) = -n\frac{\cos(n+1)\chi}{R^{n+1}}$$

and that

$$\frac{d}{dz}(R^n\cos n\chi) = -nR^{n-1}\sin(n-1)\chi$$

and

$$\frac{d}{dz}\left(\frac{\cos n\chi}{R^n}\right) = -\frac{n\sin(n+1)\chi}{R^{n+1}}$$

Let us write

$$I_1 = \int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\sin\theta\cos\phi)}}$$

I_1 has been expanded as a function of R, χ, and c.

The expansion of dI_1/dc may therefore be found by differentiating this by the rule

$$\frac{d}{dc} = \frac{\partial}{\partial c} + \cos\chi \frac{\partial}{\partial R} - \frac{\sin\chi}{R}\frac{\partial}{\partial\chi}.$$

It is rather simpler to use the formulæ for d/dc ($R^n \cos n\chi$) proved above.

Take, for example, the term

$$\frac{1}{c}\frac{9l+3}{64} s^3 \cos\chi.$$

This

$$= \frac{9l+3}{64}\frac{R^2}{c^4} R\cos\chi.$$

$$\frac{d}{dc}\left(\frac{9l+3}{64}\frac{R^2}{c^4}R\cos\chi\right) = \frac{(9l+3)R^2}{64c^4}\frac{d}{dc}(R\cos\chi) + R\cos\chi\frac{d}{dc}\left\{\frac{(9l+3)R^2}{64c^4}\right\}$$

$$= \frac{(9l+3)R^2}{64c^4} + \frac{R\cos^2\chi}{64c^4}\frac{\partial}{\partial R}\{(9l+3)R^2\} + \frac{R^3\cos\chi}{64}\frac{\partial}{\partial c}\left(\frac{9l+3}{c^4}\right).$$

In performing the partial differentiations with respect to R and c, it must be remembered that $l = \log 8c/R - 2$, and is a function of R and c.

Let

$$I_1 = \frac{l+2}{c} + \frac{l+1}{2}\frac{R\cos\chi}{c^2} + \left\{\frac{(2l+1)R^2}{16c^3} + \frac{(3l+2)R^2\cos 2\chi}{16c^3}\right\}$$
$$+ \left\{\frac{(9l+3)R^3\cos\chi}{64c^4} + \frac{(15l+7)R^3\cos 3\chi}{192c^4}\right\} + \&c.$$

$$\frac{dI_1}{dc} = -\frac{\cos\chi}{cR} - \frac{1+\cos 2\chi}{4c^2} + \left\{\frac{(8l-3)R\cos\chi}{32c^3} - \frac{3R\cos 3\chi}{32c^3}\right\}$$
$$+ \left\{\frac{(18l-3)R^2}{128c^4} + \frac{(18l-8)R^2\cos 2\chi}{128c^4} - \frac{5R^2\cos 4\chi}{128c^4}\right\} + \ldots$$
$$+ \frac{l+1}{2c^2} + \frac{(3l+2)R\cos\chi}{8c^3} + \left\{\frac{(9l+3)R^2}{64c^4} + \frac{(15l+7)R^2\cos 2\chi}{64c^4}\right\} + \ldots$$
$$- \frac{l+1}{c^2} - \frac{(2l+1)R\cos\chi}{2c^3} - \left\{\frac{(6l+1)R^2}{16c^4} + \frac{(9l+3)R^2\cos 2\chi}{16c^4}\right\} + \ldots,$$

$$= -\frac{\cos\chi}{cR} - \left(\frac{2l+3}{4c^2} + \frac{\cos 2\chi}{4c^2}\right) - \left\{\frac{(20l+11)R\cos\chi}{32c^3} + \frac{3R\cos 3\chi}{32c^3}\right\}$$
$$- \left\{\frac{(12l+5)R^2}{128c^4} + \frac{(12l+9)R^2\cos 2\chi}{64c^4} + \frac{5R^2\cos 4\chi}{128c^4}\right\} - \&c.$$

From this $d^2\mathrm{I}_1/dc^2$ might be obtained, and then $d^3\mathrm{I}_1/dc^3$, &c., but in the question of an attracting ring there is a special advantage in using functions of the form

$$\left(-\frac{1}{c}\frac{d}{dc}\right)^{n-1}\mathrm{I}_1.$$

Call this I_n. Thus

$$\mathrm{I}_n = \left(-\frac{1}{c}\frac{d}{dc}\right)^{n-1}\int_0^\pi \frac{d\phi}{\sqrt{\{r^2+c^2-2cr\sin\theta\cos\phi\}}}.$$

Also denote

$$\left(-\frac{1}{c}\frac{d}{dc}\right)^{n-1}\int_0^\pi \frac{\cos\phi\, d\phi}{\sqrt{\{r^2+c^2-2cr\sin\theta\cos\phi\}}}$$

by J_n.

The values of the functions are collected here for the sake of reference.

(A)

$$\mathrm{I}_1 = \frac{1}{c}\left\{l + 2 + \frac{l+1}{2}\cos\chi\, s + \left(\frac{2l+1}{16} + \frac{3l+2}{16}\cos 2\chi\right)s^2 \right.$$
$$+ \left(\frac{9l+3}{64}\cos\chi + \frac{15l+7}{192}\cos 3\chi\right)s^3$$
$$\left. + \left(\frac{108l+27}{2048} + \frac{60l+13}{768}\cos 2\chi + \frac{105l+34}{3072}\cos 4\chi\right)s^4 + \ldots\right\}$$

$$\mathrm{I}_2 = \frac{1}{c^2\mathrm{R}}\left\{\cos\chi + \left(\frac{2l+3}{4} + \frac{\cos 2\chi}{4}\right)s + \left(\frac{12l+11}{32}\cos\chi + \tfrac{3}{32}\cos 3\chi\right)s^2 \right.$$
$$\left. + \left(\frac{12l+5}{128} + \frac{12l+9}{64}\cos 2\chi + \tfrac{5}{128}\cos 4\chi\right)s^3 + \ldots\right\}$$

$$\mathrm{I}_3 = \frac{1}{c^3\mathrm{R}^2}\left\{\cos 2\chi + \left(\tfrac{9}{4}\cos\chi + \tfrac{1}{4}\cos 3\chi\right)s \right.$$
$$\left. + \left(\frac{36l+51}{32} + \tfrac{3}{4}\cos 2\chi + \tfrac{3}{32}\cos 4\chi\right)s^2 + \ldots\right\}$$

$$\mathrm{I}_4 = \frac{2!}{c^4\mathrm{R}^3}\left\{\cos 3\chi + \left(\tfrac{5}{2}\cos 2\chi + \tfrac{1}{4}\cos 4\chi\right)s + \ldots\right\}$$

$$\mathrm{I}_5 = \frac{3!}{c^5\mathrm{R}^4}\left\{\cos 4\chi + \ldots\right\}.$$

(B)

$$\frac{d\mathrm{I}_1}{dz} = -\frac{1}{c\mathrm{R}}\left\{\sin\chi + \tfrac{1}{4}\sin 2\chi\, s + \left(\frac{4l+5}{32}\sin\chi + \tfrac{3}{32}\sin 3\chi\right)s^2 \right.$$
$$+ \left(\frac{3l+3}{32}\sin 2\chi + \tfrac{5}{128}\sin 4\chi\right)s^3$$
$$\left. + \left(\frac{6l+3}{256}\sin\chi + \frac{120l+101}{2048}\sin 3\chi + \tfrac{35}{2048}\sin 5\chi\right)s^4 + \ldots\right\}$$

$$\frac{dI_2}{dz} = -\frac{1}{c^2R^2}\left\{\sin 2\chi + \left(\tfrac{3}{4}\sin\chi + \tfrac{1}{4}\sin 3\chi\right)s + \left(\tfrac{3}{8}\sin 2\chi + \tfrac{3}{32}\sin 4\chi\right)s^2 + \left(\frac{12l+13}{64}\sin\chi + \tfrac{27}{128}\sin 3\chi + \tfrac{5}{128}\sin 5\chi\right)s^3 + \dots\right\}$$

$$\frac{dI_3}{dz} = -\frac{2\,!}{c^3R^3}\left\{\sin 3\chi + \left(\tfrac{5}{4}\sin 2\chi + \tfrac{1}{4}\sin 4\chi\right)s + \left(\tfrac{15}{16}\sin\chi + \tfrac{15}{32}\sin 3\chi + \tfrac{3}{32}\sin 5\chi\right)s^2 + \dots\right\}$$

$$\frac{dI_4}{dz} = -\frac{3\,!}{c^4R^4}\left\{\sin 4\chi + \left(\tfrac{7}{4}\sin 3\chi + \tfrac{1}{4}\sin 5\chi\right)s + \dots\right\}$$

$$\frac{dI_5}{dz} = -\frac{4\,!}{c^5R^5}\left\{\sin 5\chi + \dots\right\}.$$

(C)

$$J_1 = \frac{1}{c}\left\{l + \frac{l-1}{2}\cos\chi\, s + \left(\frac{6l+1}{16} + \frac{3l-4}{16}\cos 2\chi\right)s^2 + \left(\frac{33l+1}{64}\cos\chi\,\frac{15l-23}{192}\cos 3\chi\right)s^3 + \left(\frac{540l+27}{2048} + \frac{240l-23}{768}\cos 2\chi + \frac{105l-176}{3072}\cos 4\chi\right)s^4 + \dots\right\}$$

$$J_2 = \frac{1}{c^2R}\left\{\cos\chi + \left(\frac{2l-1}{4} + \tfrac{1}{4}\cos 2\chi\right)s + \left(-\frac{4l+21}{32}\cos\chi + \tfrac{3}{32}\cos 3\chi\right)s^2 + \left(\frac{12l+5}{128} - \frac{12l+19}{64}\cos 2\chi + \tfrac{5}{128}\cos 4\chi\right)s^3 + \left(\frac{18l+1}{256}\cos\chi - \frac{35l+32}{256}\cos 3\chi + \tfrac{35}{2048}\cos 5\chi\right)s^4 + \dots\right\}$$

$$J_3 = \frac{1}{c^3R^2}\left\{\cos 2\chi + \left(\tfrac{9}{4}\cos\chi + \tfrac{1}{4}\cos 3\chi\right)s + \left(\frac{52l-21}{32} + \tfrac{1}{2}\cos 2\chi + \tfrac{3}{32}\cos 4\chi\right)s^2 + \left(-\frac{20l+127}{64}\cos\chi + \tfrac{21}{128}\cos 3\chi + \tfrac{5}{128}\cos 5\chi\right)s^3 + \dots\right\}$$

$$J_4 = \frac{2!}{c^4R^3}\left\{\cos 3\chi + \left(\tfrac{5}{2}\cos 2\chi + \tfrac{1}{4}\cos 4\chi\right)s + \left(\tfrac{81}{16}\cos\chi + \tfrac{21}{32}\cos 3\chi + \tfrac{3}{32}\cos 5\chi\right)s^2 + \dots\right\}$$

$$J_5 = \frac{3!}{c^5R^4}\left\{\cos 4\chi + \left(\tfrac{35}{12}\cos 3\chi + \tfrac{1}{4}\cos 5\chi\right)s + \dots\right\}$$

$$J_6 = \frac{4!}{c^6R^5}\left\{\cos 5\chi + \dots\right\}.$$

(D)

$$\frac{dJ_1}{dz} = -\frac{1}{cR}\left\{\sin\chi + \tfrac{1}{4}\sin 2\chi\, s + \left(-\frac{12l+11}{32}\sin\chi + \tfrac{3}{32}\sin 3\chi\right)s^2 + \left(-\frac{9l+5}{32}\sin 2\chi + \tfrac{5}{128}\sin 4\chi\right)s^3 + \left(-\frac{30l+9}{256}\sin\chi - \frac{360l+123}{2048}\sin 3\chi + \tfrac{35}{2048}\sin 5\chi\right)s^4 + \ldots\right\}$$

$$\frac{dJ_2}{dz} = -\frac{1}{c^2R^2}\left\{\sin 2\chi + (\tfrac{3}{4}\sin\chi + \tfrac{1}{4}\sin 3\chi)\, s + (\tfrac{1}{8}\sin 2\chi + \tfrac{3}{32}\sin 4\chi)\, s^2 + \left(-\frac{36l+31}{64}\sin\chi + \tfrac{3}{128}\sin 3\chi + \tfrac{5}{128}\sin 5\chi\right) s^3 + \ldots\right\}$$

$$\frac{dJ_3}{dz} = -\frac{2!}{c^3R^3}\{\sin 3\chi + (\tfrac{5}{4}\sin 2\chi + \tfrac{1}{4}\sin 4\chi)\, s + (\tfrac{17}{16}\sin\chi + \tfrac{11}{32}\sin 3\chi + \tfrac{3}{32}\sin 5\chi)\, s^2 + \ldots\}$$

$$\frac{dJ_4}{dz} = -\frac{3!}{c^4R^4}\{\sin 4\chi + (\tfrac{7}{4}\sin 3\chi + \tfrac{1}{4}\sin 5\chi)\, s + \ldots\}$$

$$\frac{dJ_5}{dz} = -\frac{4!}{c^5R^5}\{\sin 5\chi + \ldots\}.$$

The following formulæ are useful for questions involving the stream-line function. They are obtained by multiplying the expansions just given for J_1, J_2, &c., by ϖ or $c(1 - s\cos\chi)$.

(E)

$$J_1\varpi = l - \frac{l+1}{2}\cos\chi\, s + \left(\frac{2l+5}{16} - \frac{l}{16}\cos 2\chi\right)s^2 + \left(\frac{3l+5}{64}\cos\chi - \frac{3l-1}{192}\cos 3\chi\right)s^3 + \left(\frac{12l+11}{2048} + \frac{12l+17}{768}\cos 2\chi - \frac{15l-8}{3072}\cos 4\chi\right)s^4 + \ldots$$

$$J_2\varpi = \frac{1}{cR}\left\{\cos\chi + \left(\frac{2l-3}{4} - \tfrac{1}{4}\cos 2\chi\right)s + \left(-\frac{20l+17}{32}\cos\chi - \tfrac{1}{32}\cos 3\chi\right)s^2 + \left(\frac{20l+47}{128} - \frac{8l+1}{64}\cos 2\chi - \tfrac{1}{128}\cos 4\chi\right)s^3 + \left(\frac{18l+29}{256}\cos\chi - \frac{11l+1}{256}\cos 3\chi - \tfrac{5}{2048}\cos 5\chi\right)s^4 + \ldots\right\}$$

$$J_3\varpi = \frac{1}{c^2R^2}\left\{\cos 2\chi + (\tfrac{7}{4}\cos\chi - \tfrac{1}{4}\cos 3\chi)\, s + \left(\frac{52l-57}{32} - \tfrac{3}{4}\cos 2\chi - \tfrac{1}{32}\cos 4\chi\right)s^2 + \left(-\frac{124l+101}{64}\cos\chi - \tfrac{17}{128}\cos 3\chi - \tfrac{1}{128}\cos 5\chi\right)s^3 + \ldots\right\}$$

$$J_4\varpi = \frac{2!}{c^3R^3}\{\cos 3\chi + (2\cos 2\chi - \tfrac{1}{4}\cos 4\chi)\,s + (\tfrac{61}{16}\cos\chi - \tfrac{23}{32}\cos 3\chi - \tfrac{1}{32}\cos 5\chi)\,s^2 + \ldots\}$$

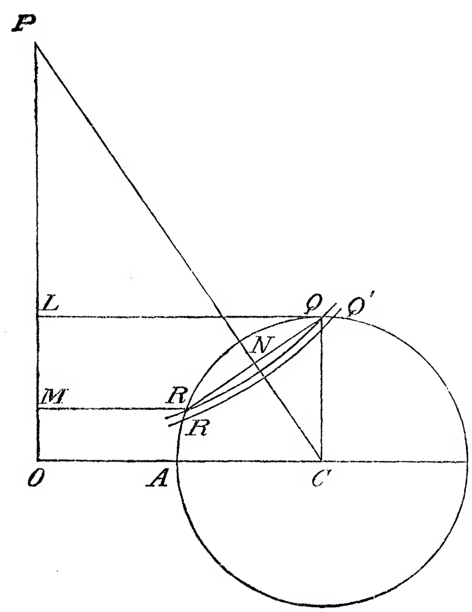
$$J_5\varpi = \frac{3!}{c^4R^4}\{\cos 4\chi + (\tfrac{29}{12}\cos 3\chi - \tfrac{1}{4}\cos 5\chi)\,s + \ldots\}$$

$$J_6\varpi = \frac{4!}{c^5R^5}\{\cos 5\chi + \ldots\}.$$

These five sets of formulæ will be referred to as (A), (B), (C), (D), (E).

Section II.—*The Potential of an Anchor Ring at an External Point.*

§ 4. The potential of an anchor ring at a point on its axis may be easily found in several ways. One simple method is to divide the ring into elements by spheres, having the given point as centre.



Let QAR be a circle which by revolution round OP generates an anchor ring. Let C be its centre, and let OC be perpendicular to OP.

With centre P describe circles, dividing the circle QAR into elements; let QR and Q′R′ be two of these circles.

By revolution of the figure round OP we obtain an anchor ring divided into elements.

Let

$$AC = a. \qquad OC = c. \qquad PC = R. \quad PQ = \rho.$$
$$\angle\, PCQ = \psi \qquad \angle\, OCP = \alpha.$$

Then

$$\rho^2 = \mathrm{R}^2 + a^2 - 2a\mathrm{R}\cos\psi.$$

Therefore

$$\rho\, d\rho = a\mathrm{R}\sin\psi\, d\psi.$$

The volume formed by the revolution of QRR′Q′ is

$$2\pi\rho\, d\rho\,.\,\mathrm{LM}.$$

The potential of this at P

$$= \frac{2\pi\rho\, d\rho\,.\,\mathrm{LM}}{\rho}$$

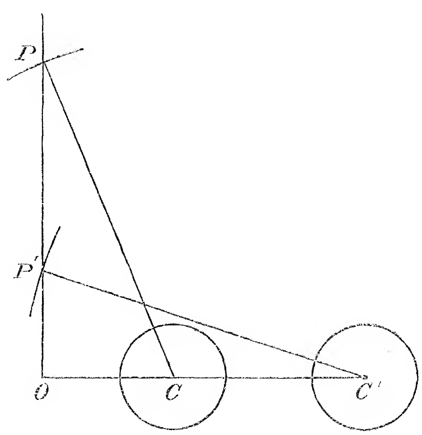
$$= 2\pi\frac{a\mathrm{R}\sin\psi\,.\,d\psi 2a\sin\psi\,.\cos\alpha}{\sqrt{\{\mathrm{R}^2 + a^2 - 2a\mathrm{R}\cos\psi\}}}$$

$$= 4\pi a^2 c\frac{\sin^2\psi\, d\psi}{\sqrt{\{\mathrm{R}^2 - 2a\mathrm{R}\cos\psi + a^2\}}}.$$

Therefore the potential of the whole ring

$$= 2\frac{\mathrm{M}}{\pi}\int_0^\pi \frac{\sin^2\psi\, d\psi}{\sqrt{\{\mathrm{R}^2 - 2a\mathrm{R}\cos\psi + a^2\}}},$$

where M is the mass of the ring.



Let there be two anchor rings of different radii, but of equal generating circles, having the same axis and centre.

Let C and C′ be the centres of the generating circles.

Let P and P′ be points on the axis such that CP = C′P′.

If the densities of the rings be inversely proportional to the distance of the centres of their generating circles from the axis of revolution, that is, if the densities be made such that the masses of the rings are equal, then the above formula shows that the potential of the C ring at P is equal to the potential of the C′ ring at P′.

§ 5. The integral

$$\int_0^{\pi} \frac{\sin^2 \psi \, d\psi}{\sqrt{\{R^2 - 2aR\cos\psi + a^2\}}}$$

may be reduced to elliptic functions.

Let it be called I. Then

$$I = -\int_0^{\pi} \frac{\sin\psi \, d(\cos\psi)}{\sqrt{\{R^2 - 2aR\cos\psi + a^2\}}}$$

$$= -\frac{1}{aR}\int_0^{\pi} \cos\psi \sqrt{\{R^2 - 2aR\cos\psi + a^2\}}\, d\psi$$

$$= -\frac{1}{aR}\int_0^{\pi} \frac{(R^2 + a^2)\cos\psi - 2aR(1 - \sin^2\psi)}{\sqrt{\{R^2 - 2aR\cos\psi + a^2\}}}\, d\psi.$$

Therefore

$$3I^* = 2\int_0^{\pi} \frac{d\psi}{\sqrt{\{R^2 - 2aR\cos\psi + a^2\}}} - \left(\frac{R}{a} + \frac{a}{R}\right)\int_0^{\pi} \frac{\cos\psi \, d\psi}{\sqrt{\{R^2 - 2aR\cos\psi + a^2\}}}.$$

Writing α for a/R,

$$I = \frac{2}{3R}\int_0^{\pi} \frac{d\psi}{\sqrt{\{1 - 2\alpha\cos\psi + \alpha^2\}}} - \frac{1}{3R}\left(\alpha + \frac{1}{\alpha}\right)\int_0^{\pi} \frac{\cos\psi \, d\psi}{\sqrt{\{1 - 2\alpha\cos\psi + \alpha^2\}}}.$$

These integrals may be transformed by putting

$$\sin(\phi - \psi) = \alpha \sin\phi.$$

This gives

$$d\psi = d\phi\left\{1 - \frac{\alpha\cos\phi}{\sqrt{\cos(\phi - \psi)}}\right\}$$

$$= d\phi\left\{1 - \frac{\alpha\cos\phi}{\sqrt{(1 - \alpha^2\sin^2\phi)}}\right\}.$$

Also

$$\sin\phi(\cos\psi - \alpha) = \cos\phi\sin\psi.$$

Therefore

$$\sin^2\phi(\cos\psi - \alpha)^2 = \cos^2\phi\sin^2\psi,$$

or

$$\cos^2\psi - 2\alpha\cos\psi + \alpha^2 = \cot^2\phi - \cot^2\phi\cos^2\psi.$$

Therefore

$$\cos^2\psi\,\mathrm{cosec}^2\phi - 2\alpha\cos\psi + \alpha^2 = \cot^2\phi,$$

or

$$(\cos\psi - \alpha\sin^2\phi)^2 = \cos^2\phi - \alpha^2\sin^2\phi + \alpha^2\sin^4\phi$$
$$= \cos^2\phi - \alpha^2\sin^2\phi\cos^2\phi.$$

* This method of reduction was given by one of the Referees.

Therefore

$$\cos\psi = \alpha\sin^2\phi + \cos\phi\sqrt{(1-\alpha^2\sin^2\phi)}.$$

Therefore

$$1 - 2\alpha\cos\psi + \alpha^2 = 1 + \alpha^2 - 2\alpha^2\sin^2\phi - 2\alpha\cos\phi\sqrt{(1-\alpha^2\sin^2\phi)}$$
$$= [\sqrt{(1-\alpha^2\sin^2\phi)} - \alpha\cos\phi]^2.$$

Therefore

$$\sqrt{(1-2\alpha\cos\psi+\alpha^2)} = \{\sqrt{(1-\alpha^2\sin^2\phi)} - \alpha\cos\phi\}.$$

Thus

$$\frac{d\psi}{\sqrt{(1-2\alpha\cos\psi+\alpha^2)}} = \frac{d\phi}{\sqrt{(1-\alpha^2\sin^2\phi)}}$$

and

$$\frac{\cos\psi\, d\psi}{\sqrt{(1-2\alpha\cos\psi+\alpha^2)}} = \frac{\alpha\sin\phi + \cos\phi\sqrt{(1-\alpha^2\sin^2\phi)}}{\sqrt{(1-\alpha^2\sin^2\phi)}}\, d\phi,$$

giving

$$\int_0^\pi \frac{d\psi}{\sqrt{(1-2\alpha\cos\psi+\alpha^2)}} = 2\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{(1-\alpha^2\sin^2\phi)}} = 2\mathrm{F}$$

$$\int_0^\pi \frac{\cos\psi\, d\psi}{\sqrt{(1-2\alpha\cos\psi+\alpha^2)}} = 2\alpha\int_0^{\frac{\pi}{2}} \frac{\sin^2\phi\, d\phi}{\sqrt{(1-\alpha^2\sin^2\phi)}} = \frac{2}{\alpha}(\mathrm{F}-\mathrm{E}).$$

Therefore

$$\int_0 \frac{\sin^2\psi\, d\psi}{\sqrt{\{1-2\alpha\cos\psi+\alpha^2\}}} = \frac{2}{3}\left\{4\mathrm{F} - \frac{2}{\alpha}\left(\alpha+\frac{1}{\alpha}\right)(\mathrm{F}-\mathrm{E})\right\}$$
$$= \frac{4}{3}\left\{\mathrm{F}\left(1-\frac{1}{\alpha^2}\right) + \mathrm{E}\left(1+\frac{1}{\alpha^2}\right)\right\}.$$

Therefore the potential

$$\mathrm{V} = \frac{4\mathrm{M}}{3\pi\mathrm{R}}\left\{\left(1-\frac{1}{\alpha^2}\right)\mathrm{F} + \left(1+\frac{1}{\alpha^2}\right)\mathrm{E}\right\}.$$

Now

$$\mathrm{F} = \frac{\pi}{2}\left\{1 + \frac{1^2}{2^2}\alpha^2 + \frac{1^2.3^2}{2^2.4^2}\alpha^4 + \&c.\right\}$$

and

$$\mathrm{E} = \frac{\pi}{2}\left\{1 - \frac{1.1}{2^2}\alpha^2 - \frac{1^2.3}{2^2.4^2}\alpha^4 - \&c.\right\}.$$

Substituting, we find

$$\mathrm{V} = \frac{\mathrm{M}}{\mathrm{R}}\left\{1 - \frac{1}{8}\alpha^2 - \frac{1}{64}\alpha^4 - \frac{5}{1024}\alpha^6 - \&c. - 2\,\frac{1^2.3^2\ldots(2n-3)^2.(2n-1)}{2^2.4^2\ldots(2n-2)^2.(2n)^2.(2n+2)}\alpha^{2n} - \&c.\right\}$$
$$= \mathrm{M}\left\{\frac{1}{\mathrm{R}} - \frac{1}{8}\frac{a^2}{\mathrm{R}^3} - \frac{1}{64}\frac{a^4}{\mathrm{R}^5} - \frac{5}{1024}\frac{a^6}{\mathrm{R}^7} - \&c. - 2\,\frac{1^2.3^2\ldots(2n-3)^2.(2n-1)}{2^2.4^2\ldots(2n-2)^2.(2n)^2.(2n+2)}\frac{a^{2n}}{\mathrm{R}^{2n+1}} - \&c.\right\}.$$

This series is less than the series whose general term is

$$\frac{1}{2n^2}\frac{a^{2n}}{\mathrm{R}^{2n+1}},$$

and is therefore convergent if $a = \mathrm{R}$ or $a < \mathrm{R}$.

§ 6. To find the potential of an anchor ring at any external point.

When $a = 0$, that is when the generating circle of the ring is infinitesimal, the potential at the point r, θ is

$$\frac{\mathrm{M}}{\pi}\int_0^\pi \frac{d\phi}{\sqrt{\{r^2 + c^2 - 2cr\sin\theta\cos\phi\}}}.$$

When $\theta = 0$, and $r = z$, *i.e.*, for a point on the axis, the expression reduces to $\mathrm{M}/\pi\ \pi/\mathrm{R}$, *i.e.*, to M/R.

$$\left(\frac{d}{c\,dc}\right)^n \int_0^\pi \frac{d\phi}{\sqrt{\{r^2 + c^2 - 2cr\sin\theta\cos\phi\}}}$$

satisfies LAPLACE's equation, and at a point on the axis of the ring becomes

$$\left(\frac{d}{c\,dc}\right)^n \int_0^\pi \frac{d\phi}{\sqrt{(z^2 + c^2)}} = (-1)^n.\pi\, 1.\,3.\,5\,\ldots\,(2n-1)\frac{1}{(z^2+c^2)^{n+\frac{1}{2}}}$$
$$= (-1)^n.\pi\frac{1.\,3.\,5\,\ldots\,(2n-1)}{\mathrm{R}^{2n+1}}.$$

Now at any point on the axis of the ring

$$\mathrm{V} = \mathrm{M}\left\{\frac{1}{\mathrm{R}} - \frac{1}{8}\frac{a^2}{\mathrm{R}^3} - \frac{1}{64}\frac{a^4}{\mathrm{R}^5} - \frac{5}{1024}\frac{a^6}{\mathrm{R}^7} - \&\mathrm{c.} - 2\frac{1^2.3^2\ldots(2n-3)^2.(2n-1)}{2^2.4^2\ldots(2n-2)^2.(2n)^2.(2n+2)}\frac{a^{2n}}{\mathrm{R}^{2n+1}} - \&\mathrm{c.}\right\}.$$

Therefore, at any external point r, θ,

$$\mathrm{V} = \frac{\mathrm{M}}{\pi}\left\{\int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\sin\theta\cos\phi)}} + \frac{a^2}{8}\frac{d}{c\,dc}\int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\sin\theta\cos\phi)}}\right.$$
$$- \frac{a^4}{192}\left(\frac{d}{c\,dc}\right)^2 \int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\sin\theta\cos\phi)}} + \&\mathrm{c.}$$
$$\left. + (-1)^{n+1}\frac{2a^{2n}}{2n+2}\frac{1.\,3.\,5\ldots(2n-3)}{2^2.4^2.6^2\ldots(2n)^2}\left(\frac{d}{c\,dc}\right)^n \int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\sin\theta\cos\phi)}} + \&\mathrm{c.}\right\}.$$

For this series is finite at all external points, satisfies LAPLACE's equation, vanishes at infinity, and agrees with the value of V on the axis. The series is very convergent, as is seen by the next paragraph.

§ 7. The integral

$$\left(\frac{d}{c\,dc}\right)^n \int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\sin\theta\cos\phi)}}$$

takes a simple form at points in the plane of the central circle. For, putting $\theta = \pi/2$, it becomes

$$\left(\frac{d}{c\,dc}\right)^n \int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\cos\phi)}}.$$

Now

$$\int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\cos\phi)}} = \int_0^\pi \frac{d\phi}{\sqrt{(r^2 - c^2\sin^2\phi)}}$$

when $r > c$, and

$$= \int_0^\pi \frac{d\phi}{\sqrt{(c^2 - r^2\sin^2\phi)}}$$

when $r < c$.

Therefore at points in the plane of x, y, further from the axis than the ring,

$$\mathrm{V} = \frac{\mathrm{M}}{\pi}\left\{\int_0^\pi \frac{d\phi}{\sqrt{(r^2 - c^2\sin^2\phi)}} + \frac{a^2}{8}\int_0^\pi \frac{\sin^2\phi\, d\phi}{(r^2 - c^2\sin^2\phi)^{\frac{3}{2}}} - \&\text{c.}\right.$$
$$\left. + (-1)^n \frac{1^2.3^2\ldots(2n-3)^2\,(2n-1)\,2a^{2n}}{2^2.4^2\ldots(2n-2)^2\,(2n)^2\,(2n+2)}\int_0^\pi \frac{\sin^{2n}\phi\, d\phi}{(r^2 - c^2\sin^2\phi)^{n+\frac{1}{2}}} + \&\text{c.}\right\}.$$

And at a point in the plane of xy between the axis and the ring

$$\mathrm{V} = \frac{\mathrm{M}}{\pi}\left\{\int_0^\pi \frac{d\phi}{\sqrt{(c^2 - r^2\sin^2\phi)}} - \frac{a^2}{8}\int_0^\pi \frac{d\phi}{(c^2 - r^2\sin^2\phi)^{\frac{3}{2}}} - \&\text{c.}\right.$$
$$\left. - \frac{1^2.3^2\ldots(2n-3)^2\,(2n-1)\,2a^{2n}}{2^2.4^2\ldots(2n-2)^2\,(2n)^2\,(2n+2)}\int_0^\pi \frac{d\phi}{(c^2 - r^2\sin^2\phi)^{n+\frac{1}{2}}} - \&\text{c.}\right\}.$$

[* The series in its general form must converge at about the same rate at which these two particular cases do. Their convergency is easily discussed. For

$$\int_0^\pi \frac{\sin^{2n}\phi\, d\phi}{(r^2 - c^2\sin^2\phi)^{n+\frac{1}{2}}} < \frac{\pi}{(r^2 - c^2)^{n+\frac{1}{2}}},$$

and r is $> a + c$.

Thus

$$\int_0^\pi \frac{\sin^{2n}\phi\, d\phi}{(r^2 - c^2\sin^2\phi)^{n+\frac{1}{2}}} < \frac{\pi}{a^{n+\frac{1}{2}}(2c + a)^{n+\frac{1}{2}}} < \frac{\pi}{a^{n+\frac{1}{2}}(2c)^{n+\frac{1}{2}}}.$$

Hence at all points in the plane of the central circle, beyond the ring, the series is more convergent than the series whose general term is

$$(-1)^n \frac{2a^{2n}}{2n(2n+2)}\,\frac{1}{2^n a^n c^n},$$

i.e., than the series whose n^{th} term is

$$\frac{1}{n(n+1)}\left(\frac{a}{2c}\right)^n.$$

Similarly at points within the ring, in the plane of the central circle, the series is more convergent than the series whose general term is

$$\frac{1}{n(n+1)}\left(\frac{a}{2c - a}\right)^n.]$$

* This consideration of the convergency was inserted at the suggestion of one of the Referees, August, 1892.

§ 8. For the purposes of calculation, it is convenient to transform the integrals of § 6 by LANDEN'S theorem.

$$\int_0^{\pi} \frac{d\phi}{\sqrt{\{r^2 - 2cr\sin\theta\cos\phi + c^2\}}} = 2\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\{R_1^2 - (R_1^2 - R^2)\sin^2\phi\}}},$$

$$= \frac{2}{R_1}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\left\{1 - \frac{R_1^2 - R^2}{R_1^2}\sin\phi\right\}}},$$

$$= \frac{4F}{R + R_1};$$

where

$$F = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \mu^2 \sin^2\phi}},$$

and

$$\mu = \frac{R_1 - R}{R_1 + R},$$

R and R_1 being the least and greatest distances of the point r, θ, from the circular axis of the ring.

Now,

$$\left.\begin{aligned} \frac{dF}{c\,dc} &= \frac{dF}{d\mu}\frac{d\mu}{c\,dc} = \frac{E - \mu'^2 F}{\mu\mu'^2}\frac{d\mu}{c\,dc} \\ \text{and} \quad \frac{dE}{c\,dc} &= \frac{dE}{d\mu}\frac{d\mu}{c\,dc} = \frac{E - F}{\mu}\frac{d\mu}{c\,dc} \end{aligned}\right\}.$$

[CAYLEY, 'Ell. Func.,' p. 48.]

But

$$R^2 = r^2 + c^2 - 2cr\sin\theta.$$

Therefore

$$\frac{dR}{dc} = \frac{c - r\sin\theta}{R} = \frac{4c^2 + R^2 - R_1^2}{4cR}.$$

Similarly

$$\frac{dR_1}{dc} = \frac{c + r\sin\theta}{R} = \frac{4c^2 + R_1^2 - R^2}{4cR}.$$

Therefore

$$\frac{d\mu}{dc} = \frac{(R_1 + R)\left(\frac{dR_1}{dc} - \frac{dR}{dc}\right) - (R_1 - R)\left(\frac{dR_1}{dc} + \frac{dR}{dc}\right)}{(R_1 + R)^2},$$

$$= 2\,\frac{R\frac{dR_1}{dc} - R_1\frac{dR}{dc}}{(R_1 + R)^2},$$

$$= \frac{R_1 - R}{R_1 + R}\,\frac{R^2 + R_1^2 - 4c^2}{2cRR_1},$$

$$= \frac{\mu}{c}\cos\psi;$$

where ψ is the angle between R and R_1.

Hence

$$\frac{d}{c\,dc}\int_0^\pi \frac{d\phi}{\sqrt{\{r^2 - 2cr\sin\theta\cos\phi + c^2\}}} = 4\,\frac{d}{c\,dc}\left(\frac{\mathrm{F}}{\mathrm{R}+\mathrm{R}_1}\right)$$

$$= \frac{1}{c^2}\left\{\mathrm{E}\,\frac{\mathrm{R}+\mathrm{R}_1}{\mathrm{RR}_1}\cos\psi - \frac{4\mathrm{F}}{\mathrm{R}+\mathrm{R}_1}\cos^2\frac{\psi}{2}\right\}.$$

Differentiating again we shall find

$$\left(\frac{d}{c\,dc}\right)^2\int_0^\pi \frac{d\phi}{\sqrt{\{r^2 - 2cr\sin\theta\cos\phi + c^2\}}}$$

$$= \frac{\mathrm{E}\,(\mathrm{R}+\mathrm{R}_1)}{c^4\mathrm{RR}_1}\left\{2\cos^3\psi - 4\cos\psi + \frac{2c^2}{\mathrm{RR}_1}\cos 2\psi\right\}$$

$$+ \frac{\mathrm{F}}{c^4(\mathrm{R}+\mathrm{R}_1)}\left\{5 + 8\cos\psi - \cos^2\psi - 4\cos^3\psi - \frac{4c^2}{\mathrm{RR}_1}\cos 2\psi\right\}.$$

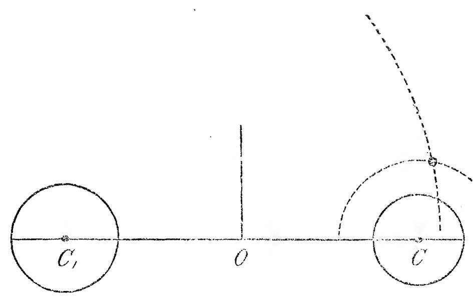
Therefore, at any external point,

$$\mathrm{V} = \frac{4\mathrm{F}}{\mathrm{R}+\mathrm{R}_1}\left\{1 - \tfrac{1}{8}\frac{a^2}{c^2}\cos^2\frac{\psi}{2} - \tfrac{1}{768}\frac{a^4}{c^4}\left[5 + 8\cos\psi - \cos^2\psi - 4\cos^3\psi - \frac{4c^2}{\mathrm{RR}_1}\cos 2\psi\right] + \&c.\right\}$$

$$+ \frac{\mathrm{E}\,(\mathrm{R}+\mathrm{R}_1)}{\mathrm{RR}_1}\left\{\frac{a^2}{8c^2}\cos\psi - \tfrac{1}{192}\frac{a^4}{c^4}\left[2\cos^2\psi - 4\cos\psi + \frac{2c^2}{\mathrm{RR}_1}\cos 2\psi\right] + \&c.\right\},$$

where ψ is the angle between R and R_1, and the modulus of the elliptic functions is

$$\frac{\mathrm{R}_1 - \mathrm{R}}{\mathrm{R}_1 + \mathrm{R}}.$$



I have calculated the value of V, retaining only the terms

$$\frac{4\mathrm{F}}{\mathrm{R}+\mathrm{R}_1}\left\{1 - \tfrac{1}{8}\frac{a^2}{c^2}\cos^2\frac{\psi}{2}\right\} + \frac{\mathrm{E}\,(\mathrm{R}+\mathrm{R}_1)}{\mathrm{RR}_1}\frac{a^2}{8c^2}\cos\psi,$$

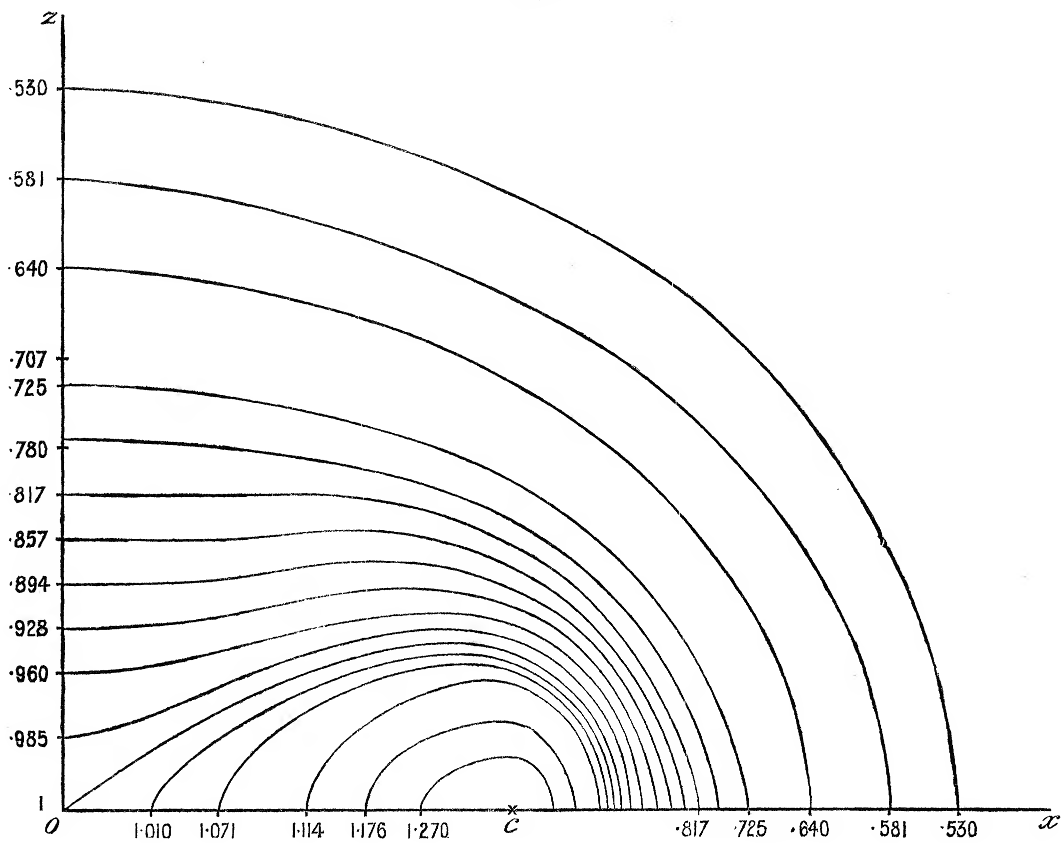
for the values

$$a = 0,\ \frac{c}{5},\ \frac{2c}{5},\ \frac{3c}{5},\ \frac{4c}{5},\ c,$$

and drawn the equipotential surfaces. The method consisted in drawing circles of known radii, with centres, C and C_1; and finding the value of V at their points of intersection.

In the most unfavourable case, where $a = c$, the value of the potential is not more than three per cent. in error at any point, the introduction of the terms in a^4/c^4 reduces the error to less than one and a half per cent. When $a < c$, the approximation is considerably nearer.

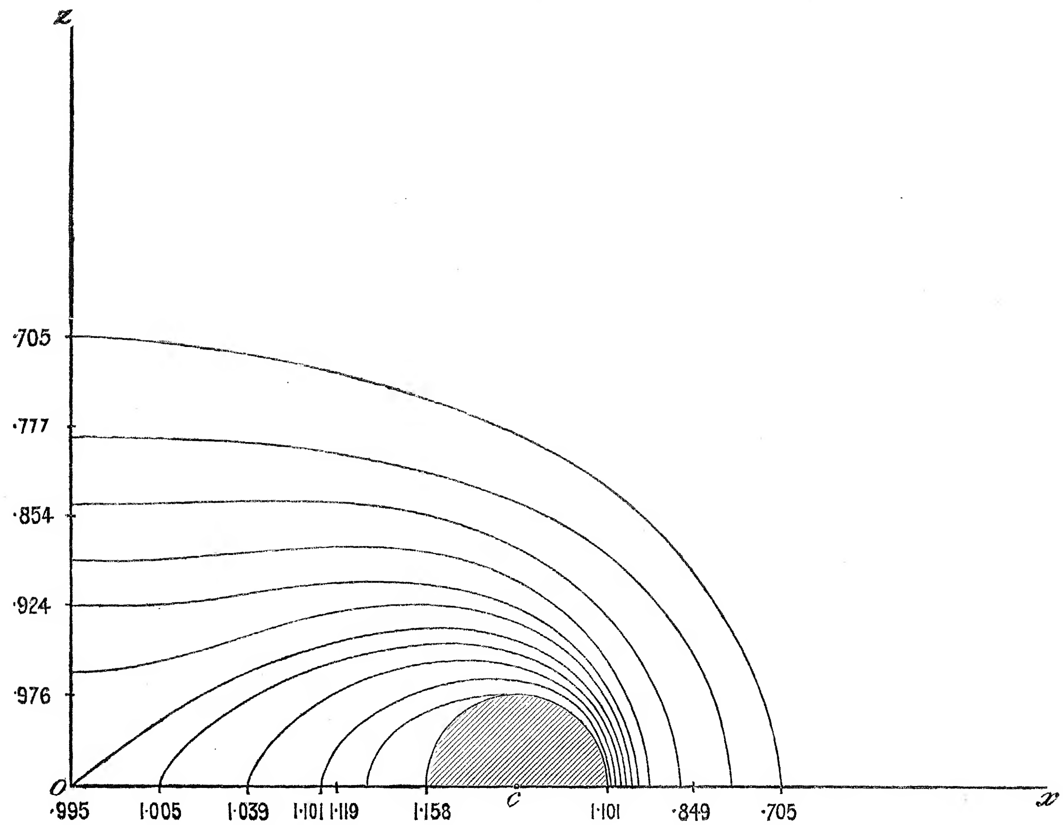
Fig. 1.



Equipotential surfaces of ring formed by the revolution of the point C about Oz.

If Mass = M and OC = c } Then $\frac{M}{c}$ is taken = 1.

Fig. 2.



Section of Equipotential surfaces of the ring formed by revolution of shaded ⊙ round Oz.

Mass = M. OC = c.

and $\frac{M}{c}$ is taken = 1.

Fig. 3.

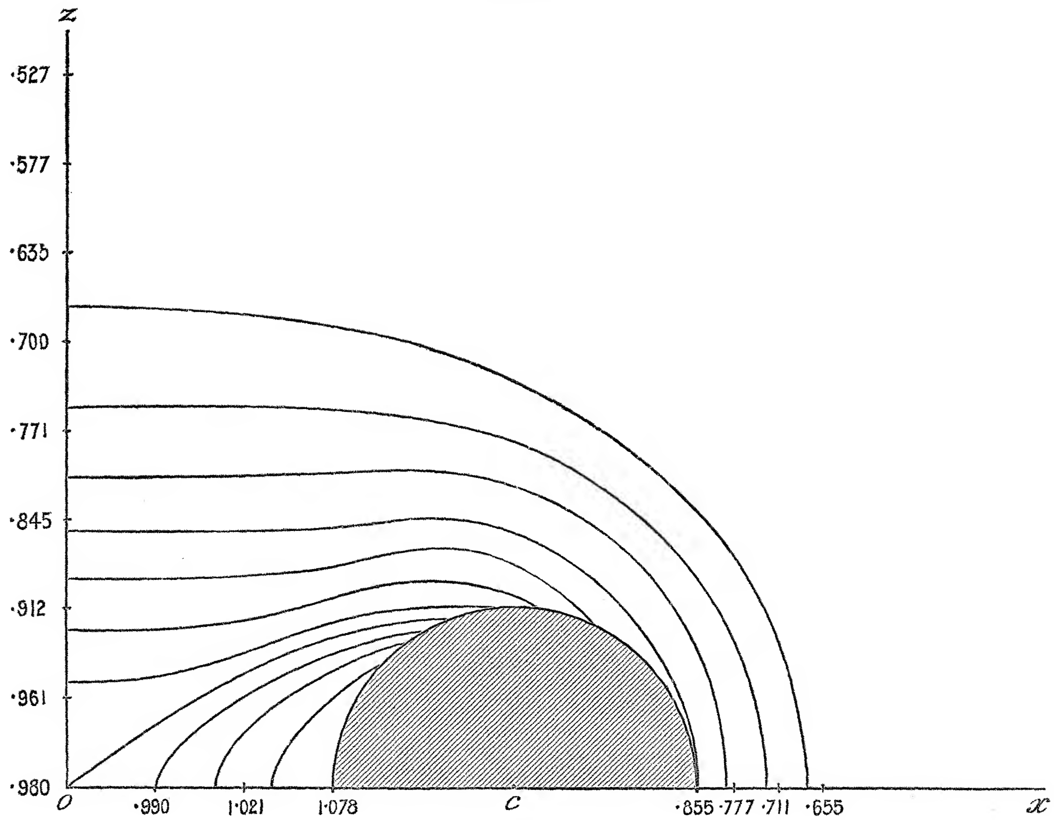

z
·527
·577
·635
·700
·771
·845
·912
·961
·980
O
·990
1·021
1·078
C
·855
·777
·711
·655
x

Fig. 4.

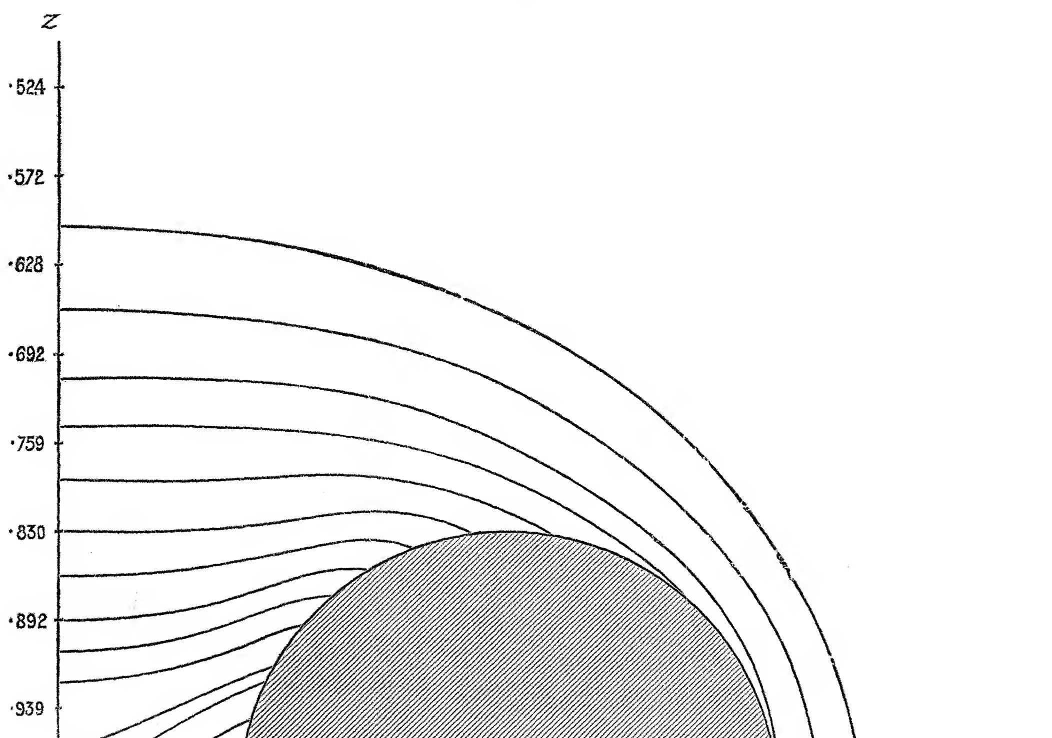

z
·524
·572
·628
·692
·759
·830
·892
·939
O
·953
·964
·998
C
·714
·659
·614
·574
·540
·510
x

Fig. 5.

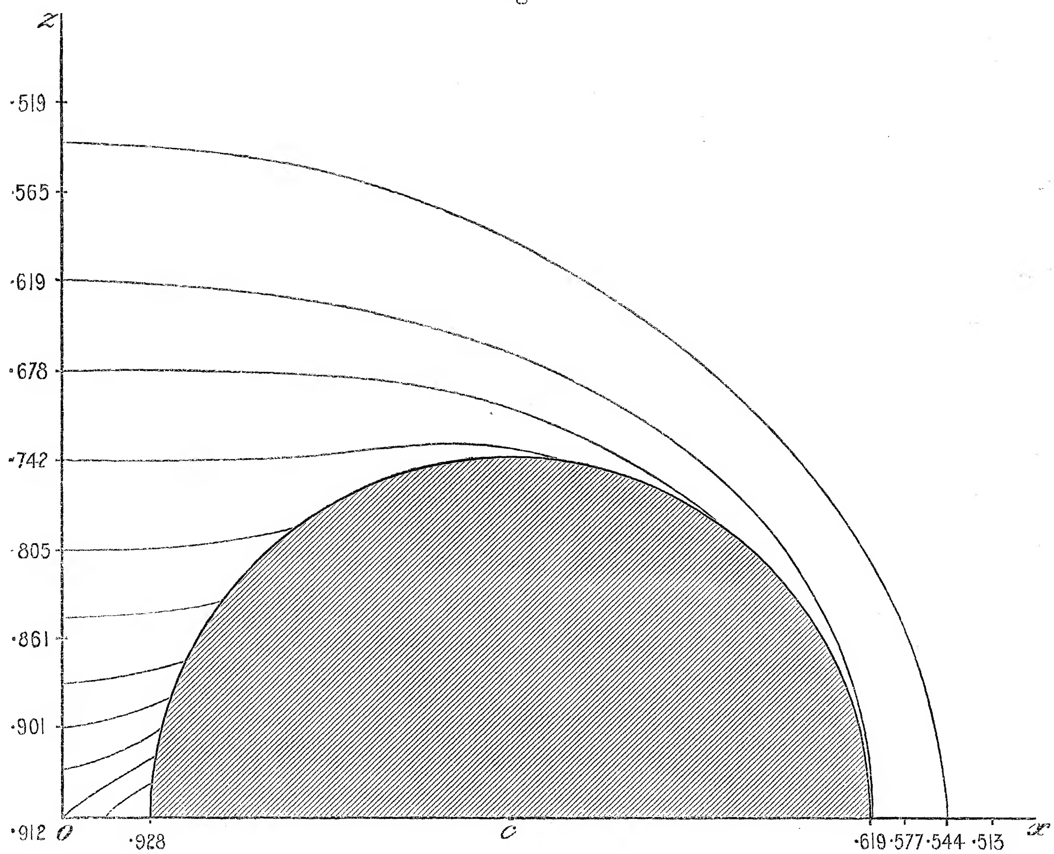
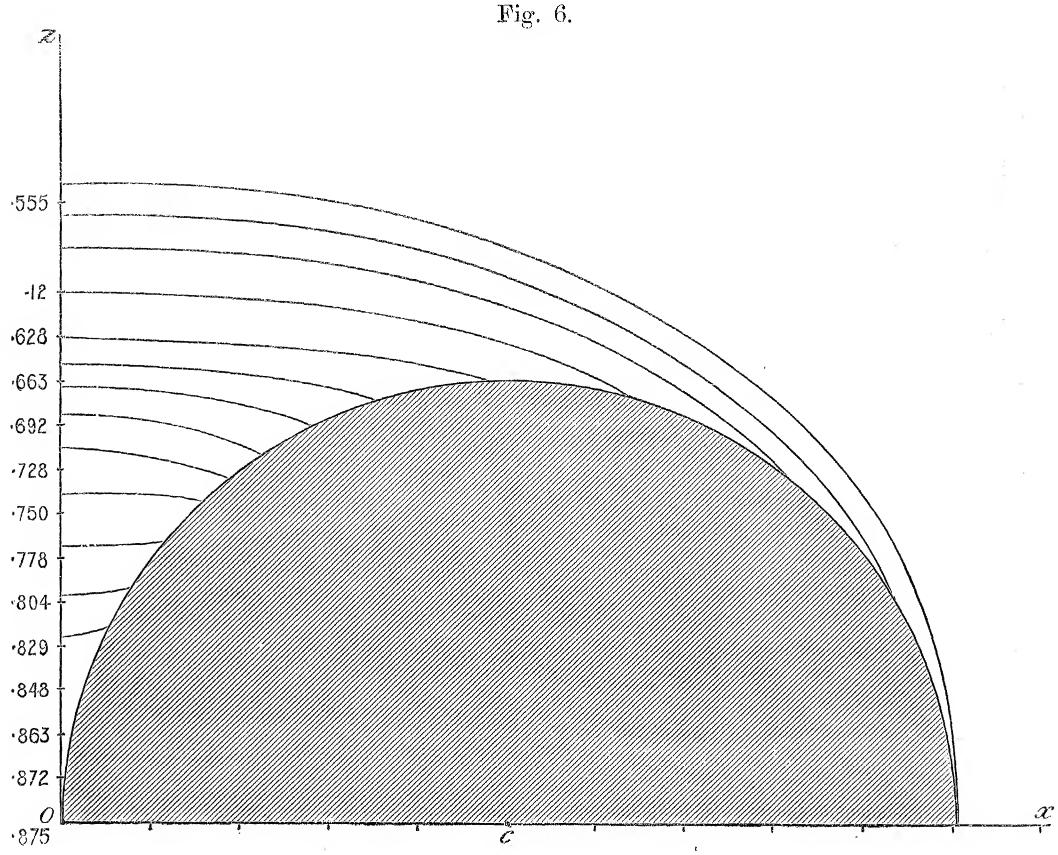

z
·519
·565
·619
·678
·742
·805
·861
·901
·912
O
·928
C
·619 ·577 ·544 ·513
x

Fig. 6.


z
·555
·12
·628
·663
·692
·728
·750
·778
·804
·829
·848
·863
·872
O
·875
C
x

Section III.—*Electric Potential of an Anchor Ring.*

§ 9. The potential at any external point may be taken to be

$$A_1 I_1 + a^2 A_2 I_2 + a^4 A_3 I_3 + \&c.$$

For this is a solution of LAPLACE'S equation, finite at all points outside the ring, and vanishing at infinity.

The constants are to be determined by the condition that at the surface of the ring the potential is the constant V_0.

Using the expansions of § 3 (A.), we have that at any external point

$$\begin{aligned} \mathrm{V}c = {} & \mathrm{A}_1 \left\{ l + 2 + \frac{2l+1}{16} s^2 + \frac{108l+27}{2048} s^4 \right\} + \mathrm{A}_2 \sigma^2 \left\{ \frac{2l+3}{4} + \frac{12l+5}{128} s^2 \right\} + \mathrm{A}_3 \sigma^4 \frac{36l+51}{32} \\ & + \cos\chi \left\{ \mathrm{A}_1 \left(\frac{l+1}{2} s + \frac{9l+3}{64} s^3 \right) + \mathrm{A}_2 \frac{\sigma^2}{s} \left(1 + \frac{12l+11}{32} s^2 \right) + \mathrm{A}_3 \sigma^4 \frac{9}{4s} \right\} \\ & + \cos 2\chi \left\{ \mathrm{A}_1 \left(\frac{3l+2}{16} s^2 + \frac{20l + 4\frac{1}{3}}{256} s^4 \right) + \mathrm{A}_2 \sigma^2 \left(\tfrac{1}{4} + \frac{12l+9}{64} s^2 \right) \right. \\ & \qquad \left. + \mathrm{A}_3 \sigma^4 \left(\frac{1}{s^2} + \tfrac{3}{4} \right) + \mathrm{A}_4 \sigma^6 \frac{5}{s^2} \right\} \\ & + \cos 3\chi \left\{ \mathrm{A}_1 \frac{5l + 2\frac{1}{3}}{64} s^3 + \mathrm{A}_2 \sigma^2 \frac{3s}{32} + \mathrm{A}_3 \sigma^4 \frac{1}{4s} + \mathrm{A}_4 \sigma^6 \frac{2}{s^3} \right\} \\ & + \cos 4\chi \left\{ \mathrm{A}_1 \frac{35l + 11\frac{1}{3}}{1024} \sigma^4 + \mathrm{A}_2 \sigma^2 \tfrac{5}{128} s^2 + \mathrm{A}_3 \sigma^4 \tfrac{3}{32} + \mathrm{A}_4 \sigma^6 \frac{1}{2s^2} + \mathrm{A}_5 \sigma^8 \frac{6}{s^4} \right\}. \end{aligned}$$

All terms have been retained which will, when R is put $= a$ or s put $= \sigma$, be of the order σ^4.

$$\frac{d\mathrm{V}}{dn} = \frac{d\mathrm{V}}{d\mathrm{R}} = \frac{d\mathrm{V}}{a\,ds},$$

s being put $= \sigma$, after differentiation.

Therefore

$$\begin{aligned} ac\frac{d\mathrm{V}}{dn} = {} & \left\{ \mathrm{A}_1 \left(-1 + \frac{\lambda}{4} \sigma^2 + \frac{27\lambda}{128} \sigma^4 \right) + \mathrm{A}_2 \left(-\frac{\sigma^2}{2} + \frac{12\lambda - 1}{64} \sigma^4 \right) - \tfrac{9}{8} \mathrm{A}_3 \sigma^4 \right\} \\ & + \sigma \cos\chi \left\{ \mathrm{A}_1 \left(\frac{\lambda}{2} + \frac{27\lambda}{64} \sigma^2 \right) + \mathrm{A}_2 \left(-1 + \frac{12\lambda - 1}{32} \sigma^2 \right) - \tfrac{9}{4} \mathrm{A}_3 \sigma^2 \right\} \\ & + \sigma^2 \cos 2\chi \left\{ \mathrm{A}_1 \frac{6\lambda+1}{16} + \frac{10\lambda - \frac{1}{3}}{32} \sigma^2 \right) + \mathrm{A}_2 \frac{12\lambda+3}{32} \sigma^2 - 2\mathrm{A}_3 - 10\mathrm{A}_4 \sigma^2 \right\} \\ & + \sigma^3 \cos 3\chi \left\{ \mathrm{A}_1 \frac{15\lambda+2}{64} + \mathrm{A}_2 \tfrac{3}{32} - \mathrm{A}_3 \tfrac{1}{4} - \mathrm{A}_4\, 6 \right\} \\ & + \sigma^4 \cos 4\chi \left\{ \mathrm{A}_1 \frac{140\lambda + 10\frac{1}{3}}{1024} + \mathrm{A}_2 \tfrac{5}{64} - \mathrm{A}_4 - 24\mathrm{A}_5 \right\}. \end{aligned}$$

The density at any point is $-\dfrac{1}{4\pi}\dfrac{dV}{dn}$.

The charge

$$
\begin{aligned}
&= -\frac{1}{4\pi}\iint \frac{dV}{dn}\,ds \\
&= -\int_0^{2\pi}\int_0^{2\pi}\frac{1}{4\pi}\frac{dV}{dn}(c - a\cos\chi)\,a\,d\chi\,d\phi \\
&= -\frac{ac}{2}\int_0^{2\pi}\frac{dV}{dn}(1 - \sigma\cos\chi)\,d\chi \\
&= \pi A_1.
\end{aligned}
$$

This value is evidently correct, for the potential at a great distance is A_1J_1 or

$$A_1\int_0^{\pi}\frac{d\phi}{\sqrt{r^2 + c^2 - 2cr\sin\theta\cos\phi}} = \frac{\pi A_1}{r}.$$

Putting $s = \sigma$ in the expression for V, we find

$$
\begin{aligned}
A_2 &= -A_1\left(\frac{\lambda + 1}{2} - \frac{12\lambda^2 + 23\lambda + 8}{64}\sigma^2\right) \\
A_3 &= -A_1\left(\frac{\lambda}{16} - \frac{12\lambda^2 + 21\lambda - 1}{256}\sigma^2\right) \\
A_4 &= -A_1\frac{\lambda - \frac{2}{3}}{128} \\
A_5 &= -A_1\frac{5\lambda - 6}{6144}.
\end{aligned}
$$

Substituting these values we find that

$$V_0 = \frac{A_1}{c}\left\{\lambda + 2 - \frac{4\lambda^2 + 8\lambda + 5}{16}\sigma^2 + \frac{192\lambda^3 + 416\lambda^2 + 448\lambda + 171}{2048}\sigma^4, \text{ \&c.}\right\}$$

and

$$
\begin{aligned}
-\frac{1}{4\pi}\frac{dV}{dn} = \frac{A_1}{4\pi ac}\Big\{&1 - \left(\frac{2\lambda + 1}{2} - \frac{24\lambda^2 + 7}{64}\sigma^2\right)\left(\frac{\sigma^2}{2} + \sigma\cos\chi\right) \\
&- \left(\frac{8\lambda - 1}{16} - \frac{36\lambda^2 + \lambda + 13}{128}\sigma^2\right)\sigma^2\cos 2\chi - \frac{16\lambda - 3}{64}\sigma^3\cos 3\chi - \frac{11(8\lambda - 1)}{1024}\sigma^4\cos 4\chi\Big\}.
\end{aligned}
$$

[The capacity (q)

$$
\begin{aligned}
&= \pi A_1/V \\
&= \pi c\Big/\left\{\lambda + 2 - \frac{4\lambda^2 + 8\lambda + 5}{16}\sigma^2 + \frac{192\lambda^3 + 416\lambda^2 + 448\lambda + 171}{2048}\sigma^4\right\}.
\end{aligned}
$$

When

$$\sigma = 1 \qquad \lambda = 2{\cdot}3820 \qquad \frac{\pi c}{q} = 4{\cdot}3820 - {\cdot}0286 + {\cdot}0003$$

$$\sigma = 2 \qquad \lambda = 1{\cdot}6889 \qquad \frac{\pi c}{q} = 3{\cdot}6889 - {\cdot}0798 + {\cdot}0022$$

$$\sigma = 3 \qquad \lambda = 1{\cdot}2834 \qquad \frac{\pi c}{q} = 3{\cdot}2834 - {\cdot}1229 + {\cdot}0073$$

$$\sigma = 4 \qquad \lambda = {\cdot}9957 \qquad \frac{\pi c}{q} = 2{\cdot}9957 - {\cdot}1691 + {\cdot}0152$$

$$\sigma = 5 \qquad \lambda = {\cdot}7726 \qquad \frac{\pi c}{q} = 2{\cdot}7726 - {\cdot}1818 + {\cdot}0260$$

The numbers are given which arise from each term, as they serve to indicate roughly the convergency of the series.

They give

$$\begin{aligned} q &= {\cdot}7216c \qquad \text{for } \sigma = 1, \\ q &= {\cdot}8700c \qquad \text{for } \sigma = 2, \\ q &= {\cdot}9917c \qquad \text{for } \sigma = 3, \\ q &= 1{\cdot}106c \qquad \text{for } \sigma = 4, \\ q &= 1{\cdot}200c \qquad \text{for } \sigma = 5, \end{aligned}$$

which agree with some figures given by Mr. HICKS in the 'Phil. Trans.,' 1881. Aug., 1892.]

Section IV.—*Motion of an Anchor Ring in an Infinite Fluid.*

§ 10. All cases of motion of an anchor ring in a fluid may be found by compounding

(i.) Linear motion parallel to the axis of the ring.
(ii.) Linear motion perpendicular to the axis of the ring.
(iii.) Rotation round a diameter of the central circle.
(iv.) Cyclic motion through the ring.

Let Φ be the velocity potential in cases (i.), (ii.), or (iii.), and Ψ be STOKES' stream-line function in cases (i.) or (iv.).

Then (a) Φ and Ψ are single-valued functions;

(b) They and their differential coefficients are finite and continuous at all points of the fluid, and vanish at infinity;

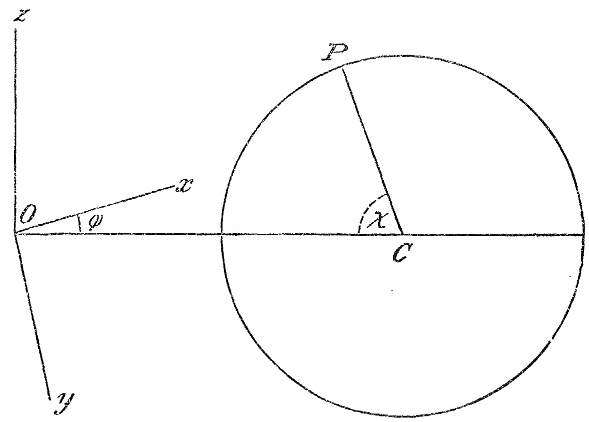
(c) Φ satisfies the equation $\nabla^2\Phi = 0$, and Ψ satisfies the equation

$$\frac{d^2\Psi}{dz^2} + \frac{d^2\Psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\Psi}{d\varpi} = 0\,;$$

(*d*) $d\Phi/dn$ has a definite value at the surface of the ring;
Ψ has a definite value at the surface of the ring.

The different cases must be considered separately.

Let the figure represent a section of the anchor ring through the axis Oz, at an azimuth ϕ.



Let P be a point on the surface and let $\angle$ OCP $= \chi$.

(i.) When the ring moves parallel to Oz, with velocity w;

$$\frac{d\Phi}{dn} = w \sin \chi.$$

(ii.) When it moves parallel to Ox with velocity u,

$$\frac{d\Phi}{dn} = -u \cos \chi \cos \phi.$$

(iii.) When it rotates round Oy with angular velocity ω_2,

$$\frac{d\Phi}{dn} = -C\omega_2 \sin \chi \cos \phi.$$

The stream function for motion parallel to Oz, with velocity w, satisfies at the surface the equation

$$\text{(i.)} \quad \Psi = C + \tfrac{1}{2} w (c - a \cos \chi)^2.$$

For the cyclic motion,

$$\text{(iv.)} \quad \Psi = C' \text{ at the surface of the ring.}$$

The formulæ of Section I. show that we may take

$$\Phi = \left\{ A_1 \frac{dI_1}{dz} + A_2 a^2 \frac{dI_2}{dz} + A_3 a^4 \frac{dI_3}{dz} + \&c. \right\} \text{ in case (i.)}$$

$$\Phi = \{A_1 J_1 + A_2 a^2 J_2 + A_3 a^4 J_3{}^2 + \&c.\} \cos \phi \text{ in case (ii.)}$$

$$\Phi = \left\{ A_1 \frac{dJ_1}{dz} + A_2 a^2 \frac{dJ_2}{dz} + A_3 a^4 \frac{dJ_3}{dz} + \&c. \right\} \cos \phi \text{ in case (iii.),}$$

and that we may also take

$$\Psi = r \sin \theta \{A_1 J_1 + A_2 a^2 J_2 + A_3 a^4 J_3 + \&c.\} \text{ in cases (i.) and (iv.).}$$

A_1, A_2, &c., are constants which must be found separately in each case.

§ 11. *Motion Parallel to the Axis of the Ring.*

Let the ring move with velocity w parallel to its axis.

The value of the normal velocity is $w \sin \chi$ at the surface of the ring.

At the surface of the ring $R = a$.

Let $\sigma = a/c$ = value of s or R/c at the surface of the ring.

Let $\lambda = \log 8c/a - 2$ = value of l or $\log 8c/R - 2$ at the surface of the ring.

Let us assume that

$$\Phi = A_1 \frac{dI_1}{dz} + A_2 a^2 \frac{dI_2}{dz} + A_3 a^4 \frac{dI_3}{dz} + \&c.$$

Then A_1, A_2, &c., are to be chosen, so that

$$\frac{d\phi}{dn} = w \sin \chi,$$

i.e.,

$$\frac{d\phi}{ds} = cw \sin \chi.$$

Differentiating the formulæ (B) of § 3, we find

$$\frac{d}{ds}\left(\frac{dI_1}{dz}\right) = \frac{1}{c^2 s^2}\left\{\sin \chi - \left(\frac{4l+1}{32} \sin \chi + \tfrac{3}{32} \sin 3\chi\right) s^2 - \left(\frac{6l+3}{32} \sin 2\chi + \tfrac{5}{64} \sin 4\chi\right) s^3 \right.$$
$$\left. - \left(\frac{18l+3}{256} \sin \chi + \frac{360l+183}{2048} \sin 3\chi + \tfrac{105}{2048} \sin 5\chi\right) s^4 - \&c.\right\}$$

$$\frac{d}{ds}\left(\frac{dI_2}{dz}\right) = \frac{1}{c^4 s^3}\left\{2 \sin 2\chi + (\tfrac{3}{4} \sin \chi + \tfrac{1}{4} \sin 3\chi)\, s \right.$$
$$\left. - \left(\frac{12l+1}{64} \sin \chi + \tfrac{27}{128} \sin 3\chi + \tfrac{5}{128} \sin 5\chi\right) s^3 - \&c.\right\}$$

$$\frac{d}{ds}\left(\frac{dI_3}{dz}\right) = \frac{2\,!}{c^6 s^4}\left\{3 \sin 3\chi + (\tfrac{5}{2} \sin 2\chi + \tfrac{1}{2} \sin 4\chi)\, s \right.$$
$$\left. + (\tfrac{15}{16} \sin \chi + \tfrac{15}{32} \sin 3\chi + \tfrac{3}{32} \sin 5\chi) s^2 + \&c.\right\}$$

$$\frac{d}{ds}\left(\frac{dI_4}{dz}\right) = \frac{3\,!}{c^8 s^5}\left\{4 \sin 4\chi + (\tfrac{21}{4} \sin 3\chi + \tfrac{3}{4} \sin 5\chi)\, s + \ldots\right\}$$

$$\frac{d}{ds}\left(\frac{dI_5}{dz}\right) = \frac{4\,!}{c^{10} s^6}\, 5 \sin 5\chi.$$

At the surface of the ring $s = \sigma = a/c$: and we have the following equations to determine the constants :—

$$\left.\begin{aligned}
&A_1\left(1 - \frac{4\lambda + 1}{32}\sigma^2 - \frac{18\lambda + 3}{256}\sigma^4\right) + A_2\left(\tfrac{3}{4}\sigma^2 - \frac{12\lambda + 1}{64}\sigma^4\right) + A_3\tfrac{15}{8}\sigma^4 = a^2cw \\
&A_1\left(-\frac{6\lambda + 3}{32}\sigma^2\right) + 2A_2 + A_3 5\sigma^2 = 0 \\
&A_1\left(-\tfrac{3}{32} - \frac{360\lambda + 183}{2048}\sigma^2\right) + A_2(\tfrac{1}{4} - \tfrac{27}{128}\sigma^2) + A_3(6 + \tfrac{15}{16}\sigma^2) + A_4\tfrac{63}{2}\sigma^2 = 0 \\
&\quad - \tfrac{5}{64}A_1 + A_3 + 24A_4 = 0 \\
&\quad - \tfrac{105}{2048}A_1 - \tfrac{5}{128}A_2 + \tfrac{3}{16}A_3 + \tfrac{9}{2}A_4 + 120A_5 = 0
\end{aligned}\right\}$$

These equations give

$$A_1 = \left\{1 + \frac{4\lambda + 1}{32}\sigma^2 + \frac{16\lambda^2 + 8\lambda - 23}{1024}\sigma^4\right\}a^2cw.$$

$$A_2 = \frac{12\lambda + 1}{128}\sigma^2a^2cw.$$

$$A_3 = \frac{1}{64}\left(1 + \frac{360\lambda + 57\frac{2}{3}}{64}\sigma^2\right)a^2cw.$$

$$A_4 = \frac{1}{384}a^2cw.$$

$$A_5 = \frac{5}{2^{14}}a^2cw.$$

Thus

$$\Phi = a^2cw\left\{\left(1 + \frac{4\lambda + 1}{32}\sigma^2 + \ldots\right)\frac{dI_1}{dz} + \frac{12\lambda + 1}{128}\sigma^2a^2\frac{dI_2}{dz} + \ldots\right\}.$$

The kinetic energy is given by the equation

$$\begin{aligned}
2T &= -\rho\iint\Phi\frac{d\Phi}{dn}dS \\
&= -\rho w\, 2\pi\int_0^{2\pi}(c - a\cos\chi)\sin\chi\,\Phi\, a d\chi \\
&= -2\pi\rho acw\int_0^{2\pi}\left(\sin\chi - \frac{\sigma}{2}\sin 2\chi\right)\Phi\, d\chi \\
&= 2\pi^2\rho\frac{ac}{c^2\sigma}w\left\{A_1\left(1 + \frac{4\lambda + 1}{32}\sigma^2 - \frac{6\lambda + 9}{256}\sigma^2\right) + A_2\sigma^2\left(\tfrac{1}{4} + \frac{12\lambda + 1}{64}\sigma^2\right) + A_3\sigma^4\tfrac{5}{8}\right\} \\
&= Mw^2\left\{1 + \frac{4\lambda + 1}{16}\sigma^2 + \frac{16\lambda^2 + 8\lambda - 23}{512}\sigma^4 + \ldots\right\},
\end{aligned}$$

where M is the mass of the fluid displaced.

§ 12. *Velocity Potential for a Ring moving at Right Angles to its Axis.*—The velocity potential Φ is of the form

$$\{A_1 J_1 + A_2 a^2 J_2 + A_3 a^4 J_3 + \ldots\} \cos\phi.$$

The constants are to be determined from the condition that the normal velocity at the surface of the ring is $-u\cos\chi\cos\phi$.

Thus

$$\frac{d\Phi}{dR} = -u\cos\chi\cos\phi$$

$$\frac{d\Phi}{ds} = -cu\cos\chi\cos\phi.$$

Now the formulæ (C) of § 3 give on differentiation

$$\begin{aligned}\frac{dJ_1}{ds} = \frac{1}{cs}\Big\{ & -1 + \frac{l-2}{2}s\cos\chi + \left(\frac{3l-1}{4} + \frac{6l-11}{16}\cos 2\chi\right)s^2 \\ & + \left(\frac{99l-30}{64}\cos\chi + \frac{15l-28}{64}\cos 3\chi\right)s^3 \\ & + \left(\frac{135l-27}{128} + \frac{240l-83}{192}\cos 2\chi + \frac{420l-809}{3072}\cos 4\chi\right)s^4 + \ldots\Big\}.\end{aligned}$$

$$\begin{aligned}\frac{dJ_2}{ds} = \frac{1}{c^3s^2}\Big\{ & -\cos\chi - \frac{s}{2} - \left(\frac{4l+}{32}\cos\chi - \tfrac{3}{32}\cos 3\chi\right)s^2 \\ & + \left(\frac{12l-1}{64} - \frac{12l+13}{32}\cos 2\chi + \tfrac{5}{64}\cos 4\chi\right)s^3 \\ & + \left(\frac{54l-15}{256}\cos\chi - \frac{105l+61}{256}\cos 3\chi + \tfrac{105}{2048}\cos 5\chi\right)s^4 + \ldots\Big\}.\end{aligned}$$

$$\begin{aligned}\frac{dJ_3}{ds} = \frac{1}{c^5s^3}\Big\{ & -2\cos 2\chi - \left(\tfrac{9}{4}\cos\chi + \tfrac{1}{4}\cos 3\chi\right)s - \tfrac{13}{8}s^2 \\ & + \left(-\frac{20l+107}{64}\cos\chi + \tfrac{21}{128}\cos 3\chi + \tfrac{5}{128}\cos 5\chi\right)s^3 + \ldots\Big\}.\end{aligned}$$

$$\begin{aligned}\frac{dJ_4}{ds} = \frac{1}{c^7s^4}\{ & -6\cos 3\chi - (10\cos 2\chi + \cos 4\chi)s \\ & - (\tfrac{81}{8}\cos\chi + \tfrac{21}{16}\cos 3\chi + \tfrac{3}{16}\cos 5\chi) + \ldots\}.\end{aligned}$$

$$\frac{dJ_5}{ds} = \frac{1}{c^9s^5}\{-24\cos 4\chi - (\tfrac{105}{2}\cos 3\chi + \tfrac{9}{2}\cos 5\chi)s - \ldots\}.$$

$$\frac{dJ_6}{ds} = \frac{1}{c^{11}s^6}\{-120\cos 5\chi - \ldots\}.$$

Therefore, the equations giving A_1, A_2, &c., are

$$\left.\begin{aligned}
&A_1\left(-1+\frac{3\lambda-1}{4}\sigma^2+\frac{135\lambda-27}{128}\sigma^4\right)+A_2\sigma^2\left(-\tfrac{1}{2}+\frac{12\lambda-1}{64}\sigma^2\right)-A_3\tfrac{13}{8}\sigma^4=0.\\
&A_1\left(\frac{\lambda-2}{2}+\frac{99\lambda-30}{64}\sigma^2\right)+A_2\left(-1-\frac{4\lambda+17}{32}\sigma^2+\frac{54\lambda-15}{256}\sigma^4\right)\\
&\qquad\qquad+A_3\sigma^2\left(-\tfrac{9}{4}-\frac{20\lambda+107}{64}\sigma^2\right)-\tfrac{81}{8}A_4\sigma^4=-c^2u.\\
&A_1\left(\frac{6\lambda-11}{16}+\frac{240\lambda-83}{192}\sigma^2\right)+A_2\sigma^2\left(-\frac{12\lambda+13}{32}\right)-2A_3-10\sigma^2A_4=0.\\
&A_1\frac{15\lambda-28}{64}+A_2\left(+\tfrac{3}{32}-\frac{105\lambda+61}{256}\sigma^2\right)+A_3\left(-\tfrac{1}{4}+\tfrac{21}{128}\sigma^2\right)\\
&\qquad\qquad+A_4\left(-6-\tfrac{21}{16}\sigma^2\right)-\tfrac{105}{2}\sigma^2A_5=0.\\
&A_1\frac{420\lambda-809}{3072}+\tfrac{5}{64}A_2-A_4-24A_5=0,\\
&\qquad\tfrac{105}{2048}A_2+\tfrac{5}{128}A_3-\tfrac{3}{16}A_4-\tfrac{9}{2}A_5-120A_6=0.
\end{aligned}\right\}$$

These equations give

$$A_1=-\frac{\sigma^2}{2}\left(1-\frac{\sigma^2}{4}\right)c^2u.$$

$$A_2=\left(1-\frac{12\lambda+1}{32}\sigma^2+\frac{48\lambda^2+344\lambda+159}{1024}\sigma^4\right)c^2u.$$

$$A_3=-\frac{18\lambda+7}{64}\sigma^2c^2u.$$

$$A_4=\left(\frac{1}{64}-\frac{24\lambda+25}{384}\sigma^2\right)c^2u.$$

$$A_5=\frac{1}{384}c^2u.$$

$$A_6=\frac{5}{2^{14}}c^2u.$$

The kinetic energy is given by the equation

$$\begin{aligned}
2T&=-\iint\Phi\frac{d\Phi}{dn}dS\\
&=\int_0^{2\pi}\int_0^{2\pi}\{A_1J_1+A_2a^2J+\ldots\}\cos\phi\,u\cos\chi\cos\phi\,(c-a\cos\chi)\,a\,d\chi\,d\phi\\
&=\pi acu\int_0^{2\pi}\{A_1J_1+\ldots\}\left\{\cos\chi-\frac{\sigma}{2}-\frac{\sigma}{2}\cos2\chi\right\}d\chi.
\end{aligned}$$

This is evaluated by picking out the coefficients of $\cos\chi$, &c., in the expansions of J_1, J_2, &c., given in § 3 (C), and

$$= \pi^2 au \left\{ A_1 \left(-\frac{\lambda+1}{2}\sigma + \frac{3\lambda+5}{64}\sigma^3 \right) + A_2 \left(\sigma - \frac{20\lambda+17}{32}\sigma^3 + \frac{18\lambda+29}{256}\sigma^5 \right) + A_3 \tfrac{7}{4}\sigma^3 + A_4 \tfrac{61}{8}\sigma^5 \right\}$$

$$= \frac{Mu^2}{2} \left\{ 1 - \frac{12\lambda+5}{16}\sigma^2 + \frac{144\lambda^2+24\lambda+57}{512}\sigma^4 \right\}.$$

§ 13. *Ring rotating about a Line through its Centre in the Plane of its Central Circle.*—The velocity potential Φ may be taken to be of the form

$$\left\{ A_1 \frac{dJ_1}{dz} + A_2 a^2 \frac{dJ_2}{dz} + \ldots \right\} \cos\phi.$$

The constants A_1, A_2, &c., are determined by the boundary condition

$$\frac{d\Phi}{dR} = -c\omega_2 \sin\chi \cos\phi,$$

or

$$\frac{d\Phi}{ds} = -c^2\omega_2 \sin\chi \cos\phi.$$

Differentiating the formulæ (D) of § 3,

$$\frac{d}{ds}\left(\frac{dJ_1}{dz}\right) = \frac{1}{c^2 s^2} \left\{ \sin\chi + \left(\frac{12l-1}{32}\sin\chi - \tfrac{3}{32}\sin 3\chi \right) s^2 + \left(\frac{18l+1}{32}\sin 2\chi - \tfrac{5}{64}\sin 4\chi \right) s^3 + \left(\frac{90l-3}{256}\sin\chi + \frac{1080l+9}{2048}\sin 3\chi - \tfrac{105}{2048}\sin 5\chi \right) s^4 \right\}$$

$$\frac{d}{ds}\left(\frac{dJ_2}{dz}\right) = \frac{1}{c^3 s^3} \left\{ 2\sin 2\chi + \left(\tfrac{3}{4}\sin\chi + \tfrac{1}{4}\sin 3\chi \right) s + \left(\frac{35l-5}{64}\sin\chi - \tfrac{3}{128}\sin 3\chi - \tfrac{5}{128}\sin 5\chi \right) s^3 + \ldots \right\}$$

$$\frac{d}{ds}\left(\frac{dJ_3}{dz}\right) = \frac{1}{c^4 s^4} \left\{ 6\sin 3\chi + (5\sin 2\chi + \sin 4\chi) s + \left(\tfrac{17}{8}\sin\chi + \tfrac{11}{16}\sin 3\chi + \tfrac{3}{16}\sin 5\chi \right) s^2 + \ldots \right\}$$

$$\frac{d}{ds}\left(\frac{dJ_4}{dz}\right) = \frac{1}{c^5 s^5} \left\{ 24\sin 4\chi + \left(\tfrac{63}{2}\sin 3\chi + \tfrac{9}{2}\sin 5\chi \right) s + \ldots \right\}$$

$$\frac{d}{ds}\left(\frac{dJ_5}{dz}\right) = \frac{1}{c^6 s^6}\, 120 \sin 5\chi + \ldots$$

The equations to determine A_1, A_2, &c., are, therefore,

$$\left.\begin{aligned}
&A_1\left(1+\frac{12\lambda-1}{32}\sigma^2+\frac{90\lambda-3}{256}\sigma^4\right)+A_2\sigma^2\left(\tfrac{3}{4}+\frac{36\lambda-5}{64}\sigma^2\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + A_3\sigma^4\,\tfrac{17}{8}=-a^2c^2\omega_2 \\
&A_1\,\frac{18\lambda+1}{32}\sigma^2+2A_2+5A_3\sigma^2=0 \\
&A_1\left(-\tfrac{3}{32}+\frac{1080\lambda+9}{2048}\sigma^2\right)+A_2\left(\tfrac{1}{4}-\tfrac{3}{128}\sigma^2\right)+A_3\left(6+\tfrac{11}{16}\sigma^2\right)+A_4\,\tfrac{63}{2}\sigma^2=0 \\
&-\tfrac{5}{64}A_1+A_3+24A_4=0 \\
&-\tfrac{105}{2048}A_1-\tfrac{5}{128}A_2+\tfrac{3}{16}A_3+\tfrac{9}{2}A_4+120\,A_5=0
\end{aligned}\right\}$$

These equations give

$$A_1=-\left(1-\frac{12\lambda-1}{32}\sigma^2+\frac{144\lambda^2-168\lambda+21}{1024}\sigma^4\right)a^2c^2\omega_2\,;$$

$$A_2=+\frac{36\lambda+7}{128}\sigma^2.\,a^2c^2\omega_2\,;$$

$$A_3=-\left(\tfrac{1}{64}-\frac{42\lambda+7\frac{1}{24}}{512}\sigma^2\right)a^2c^2\omega_2\,;$$

$$A_4=-\tfrac{1}{384}\,a^2c^2\omega_2\,;$$

$$A_5=-\frac{5}{2^{14}}\,a^2c^2\omega_2.$$

The kinetic energy is given by the equation

$$\begin{aligned}
2T&=-\iint\Phi\frac{d\Phi}{dn}\,dS \\
&=\int_0^{2\pi}\!\!\int_0^{2\pi}c\omega_2\sin\chi\,(c-a\cos\chi)\left(A_1\frac{dJ_1}{dz}+A_2a^2\frac{dJ_2}{dz}+\ldots\right)\cos^2\phi\,a\,d\chi \\
&=\pi ac^2\omega_2\int_0^{2\pi}\left(\sin\chi-\frac{\sigma}{2}\sin\chi\right)\left(A_1\frac{dJ_1}{dz}+\ldots\right)d\chi \\
&=\pi^2a^2c^3\omega_2{}^2\left\{\left(1-\frac{12\lambda-1}{32}\sigma^2+\frac{144\lambda^2-168\lambda+21}{1024}\sigma^4\right)\left(1-\frac{12\lambda+15}{32}\sigma^2+\frac{6\lambda+11}{256}\sigma^4\right)\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\left.-\frac{36\lambda+7}{128}\sigma^2\frac{\sigma^2}{4}+\tfrac{1}{64}\cdot\tfrac{7}{8}\sigma^4\right\} \\
&=M\frac{c^2\omega_2{}^2}{2}\left\{1-\frac{12\lambda+7}{16}\sigma^2+\frac{18\lambda^2-3\lambda+5}{64}\sigma^4-\ldots\right\}.
\end{aligned}$$

§ 14.—*Stream Line Function for a Ring moving parallel to its Axis, and for the Cyclic Motion through the Ring.*—Let the velocity of the ring be w, and the circulation be κ.

Let ψ be the stream-line function.

Then

$$\frac{d^2\psi}{dz^2} + \frac{d^2\psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\psi}{d\varpi} = 0$$

at all points of the fluid.

At the surface of the ring

$$\psi - \tfrac{1}{2}w\varpi^2 = \text{const.},$$

or

$$\psi = \mathrm{A}\kappa + wc^2\left(\mathrm{B} - \sigma\cos\chi + \frac{\sigma^2}{4}\cos 2\chi\right).$$

We may therefore assume ψ to be of the form

$$\{\mathrm{A}_1\mathrm{J}_1\varpi + \mathrm{A}_2 a^2\mathrm{J}_2\varpi + \mathrm{A}_3 a^4\mathrm{J}_3\varpi + \ldots\}.$$

Using the values of $\mathrm{J}_1\varpi$, $\mathrm{J}_2\varpi$, &c., given in § 4 (E),

The equations giving the constants A_1, A_2, &c., are

$$\left.\begin{aligned}
&\mathrm{A}_1\left(\lambda + \frac{2\lambda+5}{16}\sigma^2 + \frac{12\lambda+11}{2048}\sigma^4\right) + \mathrm{A}_2\sigma^2\left(\frac{2\lambda-3}{4} + \frac{20\lambda+47}{128}\sigma^2\right) \\
&\qquad\qquad + \mathrm{A}_3\sigma^4\frac{52\lambda-57}{32} = \mathrm{A}\kappa + \mathrm{B}wc^2 \\
&\mathrm{A}_1\left(-\frac{\lambda+1}{2} + \frac{3\lambda+5}{64}\sigma^2\right) + \mathrm{A}_2\left(1 - \frac{20\lambda+17}{32}\sigma^2\right) + \mathrm{A}_3\tfrac{7}{4}\sigma^2 = -wc^2 \\
&\mathrm{A}_1\left(-\frac{\lambda}{16} + \frac{12\lambda+17}{768}\sigma^2\right) + \mathrm{A}_2\left(-\tfrac{1}{4} - \frac{8\lambda+1}{64}\sigma^2\right) + \mathrm{A}_3\left(1 - \tfrac{3}{4}\sigma^2\right) + 2\mathrm{A}_4\sigma^2 = \frac{wc^2}{4} \\
&\mathrm{A}_1\frac{3\lambda-1}{192} - \frac{\mathrm{A}_2}{32} - \frac{\mathrm{A}_3}{4} + 2\mathrm{A}_4 = 0 \\
&-\mathrm{A}_1\frac{15\lambda-8}{3072} - \frac{\mathrm{A}_2}{128} - \frac{\mathrm{A}_3}{32} - \frac{\mathrm{A}_4}{2} + 6\mathrm{A}_5 = 0.
\end{aligned}\right\}$$

Another equation is found by using the fact that the circulation is κ.

Integrating round a circle concentric with the crosss-ection (radius $< c$), we have

$$\kappa = \int_0^{2\pi}\left(-\frac{1}{\varpi}\frac{d\psi}{d\mathrm{R}}\right)\mathrm{R}\,d\chi$$

$$= -\int_0^{2\pi}\frac{d}{d\mathrm{R}}\left\{\mathrm{A}_1\mathrm{J}_1\varpi + \mathrm{A}_2 a^2\mathrm{J}_2\varpi + \ldots\right\}\frac{\mathrm{R}\,d\chi}{c - \mathrm{R}\cos\chi}.$$

Differentiating the equations of § 3 (E),

$$\frac{d}{d\mathrm{R}}(\mathrm{J}_1\varpi) = \frac{1}{\mathrm{R}}\left\{-1 - \frac{l}{2}s\cos\chi + \left(\frac{l+2}{4} - \frac{2l-1}{16}\cos 2\chi\right)s^2\right.$$

$$\left. + \left(\frac{9l+12}{64}\cos\chi - \frac{3l-2}{64}\cos 3\chi\right)s^3 + \left(\frac{3l+2}{128} + \frac{6l+7}{96}\cos 2\chi - \frac{60l-47}{3072}\cos 4\chi\right)s^4\right\}.$$

$$\frac{d}{d\mathrm{R}}(\mathrm{J}_2\varpi) = \frac{1}{c\mathrm{R}^2}\left\{-\cos\chi - \frac{s}{2} - \left(\frac{20l-3}{32}\cos\chi + \tfrac{1}{32}\cos 3\chi\right)s^2\right.$$

$$\left. + \left(\frac{20l+37}{64} - \frac{3l-1}{16}\cos 2\chi - \tfrac{1}{64}\cos 4\chi\right)s^3 + \ldots\right\}.$$

$$\frac{d}{d\mathrm{R}}(\mathrm{J}_3\varpi) = \frac{1}{c^2\mathrm{R}^3}\{-2\cos 2\chi - (\tfrac{7}{4}\cos\chi - \tfrac{1}{4}\cos 3\chi)s - \tfrac{13}{8}s^2 \ldots\}.$$

Also

$$\frac{1}{c - \mathrm{R}\cos\chi} = \frac{1}{c}(1 - s\cos\chi)^{-1}$$

$$= \frac{1}{c}\left\{1 + \frac{s^2}{2} + \frac{3s^4}{8} + 2\cos\chi\,\frac{s}{2}\left(1 + \frac{3s^2}{4} + \frac{5s^4}{8}\right)\right.$$

$$\left. + 2\cos 2\chi\,\frac{s^2}{4}(1 + s^2) + 2\cos 3\chi\,\frac{s^3}{8} + \ldots\right\}.$$

Therefore

$$-\int_0^{2\pi}\frac{d}{d\mathrm{R}}(\mathrm{J}_1\varpi)\,\frac{\mathrm{R}\,d\chi}{c - \mathrm{R}\cos\chi} = \frac{2\pi}{c}\left\{\left(1 + \frac{s^2}{2} + \frac{3s^4}{8} + \ldots\right)\left(1 - \frac{l+2}{4}s^2 - \frac{3l+2}{128}s^4\right)\right.$$

$$\left. + \frac{s^2}{4}\left(1 + \frac{3s^2}{4}\right)\left(l - \frac{9l+12}{32}s^2\right) + \frac{2l-1}{64}s^4 + \ldots\right\}$$

$$= \frac{2\pi}{c}$$

Similarly

$$-\int_0^{2\pi}\frac{d}{d\mathrm{R}}(\mathrm{J}_2\varpi)\,\frac{\mathrm{R}\,d\chi}{c - \mathrm{R}\cos\chi} = \frac{2\pi}{c^3}$$

$$-\int_0^{2\pi}\frac{d}{d\mathrm{R}}(\mathrm{J}_2\varpi)\,\frac{\mathrm{R}\,d\chi}{c - \mathrm{R}\cos\chi} = 3\,\frac{2\pi}{c^5}$$

&c.

Therefore

$$\kappa = \frac{2\pi}{c}\left\{\mathrm{A}_1 + \frac{\mathrm{A}_2 a^2}{c^2} + 3\,\frac{\mathrm{A}_3 a^4}{c^4} + \ldots\right\}$$

$$= \frac{2\pi}{c}\{\mathrm{A}_1 + \mathrm{A}_2\sigma^2 + 3\mathrm{A}_3\sigma^4 + \text{\&c.}\}.$$

Thus we have, in addition to the equations above, for the determinations of A_1, A_2, &c.,

$$A_1 + A_2\sigma^2 + 3A_3\sigma^4 + \&c. = \frac{\kappa c}{2\pi}.$$

Solving the equations

$$A_1 = wc^2\sigma^2\left(1 + \frac{4\lambda + 1}{32}\sigma^2\right) + \frac{\kappa c}{2\pi}\left(1 - \frac{\lambda + 1}{2}\sigma^2 - \frac{4\lambda^2 + 17\lambda + 6}{64}\sigma^4\right)$$

$$A_2 = -wc^2\left(1 + \frac{4\lambda + 1}{32}\sigma^2\right) + \frac{\kappa c}{2\pi}\left(\frac{\lambda + 1}{2} + \frac{4\lambda^2 - 19\lambda - 18}{64}\sigma^2\right)$$

$$A_3 = -wc^2\sigma^2\frac{12\lambda - 1}{128} + \frac{\kappa c}{2\pi}\left(\frac{3\lambda + 2}{16} + \frac{12\lambda^2 + \lambda + 1}{256}\sigma^2\right)$$

$$A_4 = -wc^2\tfrac{1}{64} + \frac{\kappa c}{2\pi}\frac{9\lambda + 10}{384}$$

$$A_5 = -wc^2\tfrac{1}{256} + \frac{\kappa c}{2\pi}\frac{81\lambda + 56}{18432}.$$

Substituting these values we obtain

$$A = \frac{c}{2\pi}\left\{\lambda - \frac{4\lambda^2 + 8\lambda + 1}{16}\sigma^2 - \frac{2\lambda^3 + 9\lambda^2 + 6\lambda - \frac{43}{32}}{64}\sigma^4\right\}$$

$$B = \sigma^2\left\{\frac{2\lambda + 3}{4} + \frac{4\lambda^2 + 5\lambda - 2}{64}\sigma^4\right\}.$$

The kinetic energy is given by the equation

$$T = -\pi\rho\int_0^{2\pi}\left[\frac{\psi}{\omega}\cdot\frac{d\psi}{dR}\right]_{R=a} a\,d\chi$$

$$= -\pi\rho\int_0^{2\pi}\frac{A\kappa + wc^2\left(B - \sigma\cos\chi + \frac{\sigma^2}{4}\cos 2\chi\right)}{c - a\cos\chi}\frac{d\psi}{dR}\,a\,d\chi$$

$$= \pi\rho\int_0^{2\pi}\left\{A\kappa + wc^2\left(B - \tfrac{1}{2} - \frac{\sigma^2}{4}\right)\right\}\left(-\frac{1}{\varpi}\frac{d\psi}{dR}\right)a\,d\chi$$

$$- \frac{\pi\rho acw}{2}\int_0^{2\pi}(1 - \sigma\cos\chi)\frac{d\psi}{dR}\,d\chi$$

$$= \pi\rho A\kappa^2 + \pi\rho c^2\left(B - \tfrac{1}{2} - \frac{\sigma^2}{4}\right)\kappa w$$

$$+ \pi^2\rho wc\left\{A_1\left(1 - \frac{\lambda + 1}{2}\sigma^2 + \frac{3\lambda + 5}{64}\sigma^4\right) - A_2\frac{20\lambda + 17}{32}\sigma^4 + \tfrac{3}{4}A_3\sigma^4\right.$$

$$= \pi\rho\left\{A\kappa^2 + c^2\kappa w\left(B - \tfrac{1}{2} - \frac{\sigma^2}{4} + \tfrac{1}{2} - \frac{\lambda + 1}{2}\sigma^2 - \frac{4\lambda^2 + 5\lambda - 2}{64}\sigma^4\right)\right.$$

$$\left. + w^2c^3\sigma^2\left(1 + \frac{4\lambda + 1}{16}\sigma^2\right)\right\}.$$

The coefficient of κw vanishes, as it necessarily should (BASSET, 'Hydrodynamics,' vol. 1, art. 171), and the kinetic energy of the forward motion and the cyclic motion is given by

$$2\mathrm{T} = \mathrm{M}w^2\left(1 + \frac{4\lambda + 1}{16}\sigma^2 + \ldots\right)$$
$$+ \rho c\, \kappa^2\left(\lambda - \frac{4\lambda^2 + 8\lambda + 1}{16}\sigma^2 - \frac{2\lambda^3 + 9\lambda^2 + 6\lambda - \frac{43}{32}}{64}\sigma^4 + \ldots\right).$$

The first part of this result agrees, as far as it goes, with the result of § 11.

§ 15. As the cyclic motion through a ring is of considerable interest, it seems worth while to give other proofs of some of the above results.

To find the circulation we may integrate the velocity along the axis of the ring from $-\infty$ to ∞, and then along a semicircle from ∞ to $-\infty$.

$$\psi = \{\mathrm{A}_1\mathrm{J}_1\varpi + \mathrm{A}_2 a^2\,\mathrm{J}_2\varpi + \ldots\}.^*$$

The velocity of the fluid at any point on the axis is the limit of $\frac{1}{\varpi}\frac{d\psi}{d\varpi}$, where ϖ is indefinitely small.

Consider the part arising from the term $\mathrm{A}_1\mathrm{J}_1\varpi$ or

$$\mathrm{A}_1\int_0^\pi \frac{\varpi\cos\phi\, d\phi}{\sqrt{(z^2 + \varpi^2 - 2\varpi c\cos\phi + c^2)}}.$$

$$\text{It} = \text{limit of } \frac{\mathrm{A}_1}{\varpi}\frac{d}{d\varpi}\int_0^\pi \frac{\varpi\cos\phi\, d\phi}{\sqrt{(z^2 + \varpi^2 - 2\varpi c\cos\phi + c^2)}}$$

$$= \text{limit of} \int_0^\pi \frac{\mathrm{A}_1(z^2 + c^2 - \varpi c\cos\phi)}{\varpi(z^2 + \varpi^2 - 2\varpi c\cos\phi + c^2)^{\frac{3}{2}}}\cos\phi\, d\phi$$

$$= \text{limit of} \int_0^\pi \frac{\mathrm{A}_1}{\varpi(z^2 + c^2)^{\frac{1}{2}}}\left(1 - \frac{\varpi c\cos\phi}{z^2 + c^2}\right)\left(1 + \frac{3\varpi c\cos\phi}{z^2 + c^2}\right)\cos\phi\, d\phi$$

$$= \frac{\pi\mathrm{A}_1 c}{(z^2 + c^2)^{\frac{3}{2}}}.$$

We may notice that at the centre of the ring this

$$= \frac{\pi\mathrm{A}_1}{c^2}.$$

* As we are only considering the cyclic motion, only the parts of A_1, A_2, &c., which involve κ, are denoted by these letters now.

Again, the part of the velocity of the fluid arising from the term $A_1 J_1 \varpi$ in the stream function along an arc of a very great circle whose centre is at the origin, is the limit of $-\frac{1}{\varpi}\frac{d}{dr}(A_1 J_1 \varpi)$

$$= \text{limit of } -\frac{A_1}{r \sin\theta}\int_0^\pi \frac{\sin\theta\,(c^2 - cr\sin\theta\cos\phi)}{(r^2 + c^2 - 2cr\sin\theta\cos\phi)^{\frac{3}{2}}}\cos\phi\, d\phi$$

$$= \frac{\pi}{2}\frac{A_1 c \sin\theta}{r^3}.$$

Therefore, the part of the circulation rising from the term $A_1 J_1 \varpi$ of the stream function is

$$\int_{-\infty}^{\infty} \frac{\pi A_1 c}{(z^2 + c^2)^{\frac{3}{2}}}\, dz + \int_0^\pi \frac{\pi}{2}\frac{A_1 c \sin\theta}{r^3}\, r\, d\theta$$

$$= \frac{2\pi}{c} A_1.$$

Now the part arising from $A_2 a^2 J_2 \varpi_2$ or $- A_2 a^2 \frac{d}{c\,dc}(J_1 \varpi)$, is obtained by differentiation with respect to c, and

$$= A_2 a^2 \frac{2\pi}{c^3}.$$

Therefore, the circulation is given by

$$\kappa = \frac{2\pi}{c}\left\{A_1 + A_2\sigma^2 + 1.3\, A_3\sigma^4 + 1.3.5.\, A_4\sigma^6 + \ldots\right\},$$

agreeing with the result already obtained.

We also see that the velocity of the fluid at the centre of the ring is

$$\frac{\pi A_1}{c^2} - A_2 a^2 \frac{d}{c\,dc}\left(\frac{\pi}{c^2}\right) + \ldots = \frac{\pi}{c^2}\{A_1 + 2\sigma^2 A_2 + 8\sigma^4 A_3 + \ldots\}.$$

The kinetic energy of the cyclic motion may be obtained simply in the following way—

$$2T = \rho \iint \kappa \frac{d\Phi}{dn}\, dS,$$

the integral being taken over a barrier.

This

$$= \rho \iint \kappa \frac{1}{\varpi}\frac{d\psi}{d\varpi}\, \varpi\, d\varpi\, d\phi$$

$$= 2\pi\rho\kappa \int \frac{d\psi}{d\varpi}\, d\varpi$$

$$= 2\pi\rho\kappa\, (\psi_R - \psi_A),$$

where ψ_R denotes the value of ψ at the surface of the ring and ψ_A at the axis. This result applies to the cyclic motion through a circular ring of any cross-section.

We have taken ψ_A to be zero, and called ψ_R, A : so that we have $T = \pi\rho\kappa A$.

The Linear Momentum of the Cyclic Motion is parallel to the axis of the ring and

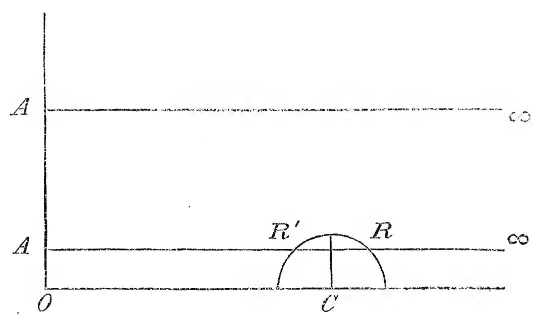
$$= \rho \iiint \frac{1}{\varpi} \frac{d\psi}{d\varpi} \varpi \, d\varpi \, dz \, d\phi :$$

the integral being taken all over the fluid.

This integral

$$= 4\pi\rho \int_0^\infty dz \int \frac{d\psi}{d\varpi} d\varpi$$

$$= 4\pi\rho \int_0^a (\psi_\infty - \psi_R + \psi_{R'} - \psi_A) \, dz + 4\pi\rho \int_a^\infty (\psi_\infty - \psi_A) \, dz.$$



We have taken $\psi_A = 0$.

ψ is constant on the ring, so that

$$\psi_{R'} - \psi_R = 0.$$

Therefore, the resultant linear momentum

$$= 4\pi\rho \int_0^\infty \psi_\infty \, dz,$$

where ψ_∞ means the value of ψ at an infinite distance from the axis.

Consider first the part of ψ_∞ arising from the term $A_1 J_1 \varpi$.

This is the limit of

$$A_1 \int_0^\pi \frac{\varpi \cos\phi \, d\phi}{\sqrt{(z^2 + c^2 - 2\varpi c \cos\phi + \varpi^2)}}$$

where ϖ is infinite.

This

$$= A_1 \int_0^\pi \frac{\varpi}{\sqrt{(\varpi^2 + z^2)}} \left\{ 1 + \frac{\varpi c \cos\phi}{\sqrt{(z^2 + c^2)}} \right\} \cos\phi \, d\phi$$

$$= A_1 \frac{\pi}{2} \frac{c\varpi^2}{(\varpi^2 + z^2)^{\frac{3}{2}}} \cdot$$

Therefore the momentum of the cyclic motion is

$$4\pi\rho\,\frac{\pi}{2}\left(\mathrm{A}_1 c - \mathrm{A}_2\frac{a^2}{c} - \mathrm{A}_3\frac{a^4}{c^3} - 1.3.\mathrm{A}_4.\frac{a^6}{c^5} + \ldots\right)$$
$$= 2\pi^2\rho c\,\{\mathrm{A}_1 - \mathrm{A}_2\,\sigma^2 - \mathrm{A}_3\,\sigma^4 - 1.3.\mathrm{A}_4\,\sigma^6 - 1.3.5.\mathrm{A}_5\,\sigma^8 + \ldots\}$$
$$= \pi^2\kappa\rho c\left\{1 - (\lambda + 1)\,\sigma^2 - \frac{4\lambda^2 + 5\lambda - 2}{32}\,\sigma^4 - \&c.\right\}.$$

§ 16. The cyclic motion might have been started by applying an impulsive pressure $\kappa\rho$ at all points of the aperture of the ring, this would produce an impulsive pressure at all points of the ring. To determine this, the velocity potential of the cyclic must be found at these points. It is easy to find an expression for the velocity potential at points *not far from the surface of the ring.*

For

$$\frac{d\Phi}{\mathrm{R}d\chi} = -\frac{1}{\varpi}\,\frac{d\psi}{d\mathrm{R}}$$

$$= -\frac{1}{\varpi}\,\frac{d}{d\mathrm{R}}\,\{\mathrm{A}_1\mathrm{J}_1\varpi + \mathrm{A}_2 a^2\,\mathrm{J}_2\varpi + \mathrm{A}_3 a^4\,\mathrm{J}_3\varpi + \ldots\}$$

$$= \frac{\mathrm{A}_1}{c\mathrm{R}\,(1 - s\cos\chi)}\left\{1 + \frac{l}{2}\,s\cos\chi - \left(\frac{l+2}{4} - \frac{2l-1}{16}\cos 2\chi\right)s^2 \right.$$
$$- \left(\frac{9l+12}{64}\cos\chi - \frac{3l-2}{64}\cos 3\chi\right)s^3$$
$$\left. - \left(\frac{3l+2}{128} + \frac{6l+7}{96}\cos 2\chi - \frac{60l-47}{3072}\cos 4\chi\right)s^4 \ldots\right\}$$

$$+ \frac{\mathrm{A}_2 a^2}{c^2\mathrm{R}^2\,(1 - s\cos\chi)}\left\{\cos\chi + \frac{s}{2} + \left(\frac{20l-3}{32}\cos\chi + \tfrac{1}{32}\cos 3\chi\right)s^2\right.$$
$$\left. - \left(\frac{20l+37}{64} - \frac{3l-1}{16}\cos 2\chi - \tfrac{1}{64}\cos 4\chi\right)s^3,\ \&c.\right\}$$

$$+ \frac{\mathrm{A}_3 a^4}{c^3\mathrm{R}^3\,(1 - s\cos\chi)}\,\{2\cos 2\chi + (\tfrac{7}{4}\cos\chi - \tfrac{1}{4}\cos 3\chi)\,s - \tfrac{13}{8}\,s^2\}$$

$$+ \frac{\mathrm{A}_4 a^6}{c^4\mathrm{R}^4\,(1 - s\cos\chi)}\,\{6\cos 3\chi + (8\cos 2\chi - \cos 4\chi)\,s\} + \&c.$$

Expanding $\dfrac{1}{1 - \sigma\cos\chi}$ in ascending powers of σ, and multiplying, $d\Phi/d\chi$ is easily expressed in cosines of multiples of χ

The term independent of χ is $\kappa/2\pi$

On integration, we obtain

$$\Phi = \text{const} + \frac{\kappa}{2\pi}\chi$$

$$+ \frac{A_1}{c}\left\{\left(\frac{l+2}{2}s + \frac{3l+2}{64}s^3\right)\sin\chi + \left(\frac{6l+7}{32}s^2 + \frac{15l+7}{384}s^4\right)\sin 2\chi + \frac{5l+4}{32}s^3\sin 3\chi + \frac{35l+20\frac{1}{12}}{1024}s^4\sin 4\chi\right\}$$

$$+ \frac{A_2\sigma^2}{cs}\left\{\left(1 + \frac{20l+37}{32}s^2\right)\sin\chi + \left(\frac{s}{4} + \frac{16l+21}{64}s^3\right)\sin 2\chi + \tfrac{3}{32}s^2\sin 3\chi + \tfrac{5}{128}s^3\sin 4\chi\right\}$$

$$+ \frac{A_3\sigma^4}{c^2s^2}\left\{\tfrac{11}{4}s\sin\chi + \left(1 + \tfrac{7}{8}s^2\right)\sin 2\chi + \tfrac{1}{4}s\sin 3\chi + \tfrac{3}{32}s^3\sin 4\chi\right\}$$

$$+ \frac{A^4\sigma^6}{c^3s^3}\left\{\tfrac{11}{2}s\sin 2\chi + 2\sin 3\chi + \frac{s}{2}\sin 4\chi\right\}$$

$$+ \frac{A_5\sigma^8}{c^4s^3}4\sin 4\chi.$$

This holds only at points not far from the surface of the ring.

At the surface of the ring $s = \sigma$. Substituting the values of the constants given in § 14, we find that here

$$\Phi = \frac{\kappa}{2\pi}\left\{\chi + \left(\frac{2\lambda+3}{2}\sigma + \frac{8\lambda^2+26\lambda+11}{64}\sigma^3\right)\sin\chi + \frac{16\lambda+15}{32}\sigma^2\sin 2\chi + \frac{57\lambda+49}{192}\sigma^3\sin 3\chi + \&\text{c.}\right\}.$$

This gives the impulsive pressure at any point on the surface of the ring.

The momentum of the cyclic motion may be deduced.

It

$$= \pi(c-a)^2\kappa\rho - \iint\Phi\sin\chi\, dS$$

$$= \pi\kappa\rho(c-a)^2 - 2\pi a\int_0^{2\pi}\Phi\sin\chi(c - a\cos\chi)\,d\chi$$

$$= \pi\kappa\rho(c-a)^2 - \frac{\kappa\rho}{2\pi}2\pi ac\int_0^{2\pi}\left(\sin\chi - \frac{\sigma}{2}\sin 2\chi\right)$$

$$\left(\chi + \left\{\frac{2\lambda+3}{2}\sigma + \frac{8\lambda^2+26\lambda+11}{64}\sigma^3\right\}\sin\chi + \frac{16\lambda+15}{32}\sigma^2\sin 2\chi\right)d\chi$$

$$= \pi\kappa\rho c^2\left\{1 - 2\sigma + \sigma^2 + \left(2\sigma - \frac{\sigma^2}{2}\right) - \sigma^2\left(\frac{2\lambda+3}{2} + \frac{8\lambda^2+26\lambda+11}{64}\sigma^2\right) + \frac{16\lambda+15}{64}\sigma^4\right\}$$

$$= \pi\kappa\rho c^2\left\{1 - (\lambda+1)\sigma^2 - \frac{4\lambda^2+5\lambda-2}{32}\sigma^4\right\},$$

which agrees with the result already obtained.

If the cross-section of the ring is very small

$$A_1 = \frac{\kappa c}{2\pi}$$

and

$$\Phi = \text{const} + \frac{\kappa}{2\pi}\left\{\chi + \left(\frac{l+2}{2}s + \frac{3l+2}{64}s^3\right)\sin\chi + \left(\frac{6l+7}{32}s^2 + \frac{15+7}{384}s^4\right)\sin 2\chi \right.$$
$$\left. + \frac{5l+4}{32}s^3\sin 3\chi + \frac{35l+20\frac{1}{12}}{1024}s^4\sin 4\chi + \ldots\right\}.$$

But in this case Φ is proportional to Ω the solid angle subtended by the ring at any point.

Hence the solid angle subtended by a circular ring at a point near it is

$$2\left\{\pi - \chi - \left(\frac{l+2}{2}s + \frac{3l+2}{64}s^3\right)\sin\chi - \left(\frac{6l+7}{32}s^2 + \frac{15l+7}{384}s^4\right)\sin 2\chi - \&c.\right\}$$

§ 17. The kinetic energy of the fluid has been calculated for the different motions of the ring. To find the kinetic energy of the solid ring, its moments of inertia round the axis, and round a perpendicular to the axis through its centre must be found

Let ρ' be the density of the ring, and M′ be the mass of the ring.

The moment of inertia round the axis is

$$\begin{aligned}\int_0^a\int_0^{2\pi} & 2\pi\rho'(c - R\cos\chi)^3 R\,dR\,d\chi \\ &= 4\pi^2\rho'\int_0^a (c^3 + \tfrac{3}{2}cR^2)R\,dR \\ &= M'(c^2 + \tfrac{3}{4}a^2).\end{aligned}$$

The moment of inertia round a perpendicular axis is

$$\begin{aligned}\int_0^a\int_0^{2\pi} & 2\pi\rho'(c - R\cos\chi)\left\{\frac{(c-R\cos\chi)^2}{2} + R^2\sin^2\chi\right\}R\,dR\,d\chi \\ &= M'\left(\frac{c^2}{2} + \tfrac{3}{8}a^2\right) + M'\,.\,\frac{a^2}{4} \\ &= M'\left(\frac{c^2}{2} + \tfrac{5}{8}a^2\right).\end{aligned}$$

§ 18. Therefore, ρ being the density of the fluid, M and M′ the masses of the fluid displaced by the ring, and of the ring, respectively, the kinetic energy is given by

$$2T = P(u^2 + v^2) + Rw^2 + A(\omega_1^2 + \omega_2^2) + C\omega_3^2 + K\kappa^2,$$

where

$$P = \frac{M}{2}\left(1 - \frac{12\lambda + 1}{16}\sigma^2 + \frac{144\lambda^2 + 24\lambda + 57}{512}\sigma^4\right) + M',$$

$$R = M\left(1 + \frac{4\lambda + 1}{16}\sigma^2 + \frac{16\lambda^2 + 8\lambda - 23}{512}\sigma^4\right) + M',$$

$$A = M\frac{c^2}{2}\left(1 - \frac{12\lambda + 7}{16}\sigma^2 + \frac{18\lambda^2 - 3\lambda + 5}{64}\sigma^4\right) + M'\frac{c^2}{2}\left(1 + \tfrac{5}{4}\sigma^2\right),$$

$$C = M'c^2\left(1 + \tfrac{3}{4}\sigma^2\right),$$

$$K = \rho c\left(\lambda - \frac{4\lambda^2 + 8\lambda + 1}{16}\sigma^2 - \frac{2\lambda^3 + 9\lambda^2 + 6\lambda - \frac{43}{32}}{64}\sigma^4\right),$$

Also the linear momentum of the cyclic motion is given by

$$Z = \pi c^2 \kappa\rho\left\{1 - (\lambda + 1)\sigma^2 - \frac{4\lambda^2 + 5\lambda - 2}{32}\sigma^4\right\}.$$

§ 19. The motion of a ring in an infinite fluid is discussed in BASSET's 'Hydrodynamics,' vol. 1, pp. 196–208.

It is there shown that if a ring be set rotating with angular velocity ω about a diameter of its circular axis, it will make complete revolutions if

$$\omega > Z\sqrt{\frac{R}{AP(R - P)}},$$

but will oscillate if

$$\omega < Z\sqrt{\frac{R}{AP(R - P)}}.$$

Putting in the values of the constants, a very fine ring will make complete revolutions if

$$\omega > \frac{\kappa}{\pi a^2}\sqrt{\frac{2\rho}{\rho + 2\rho'}},$$

and will oscillate if

$$\omega < \frac{\kappa}{\pi a^2}\sqrt{\frac{2\rho}{\rho + 2\rho'}}.$$

A possible steady motion of the ring occurs when it moves with velocity w parallel to the axis, and has also an angular velocity Ω round its axis. It is shown that this motion will be stable if

$$(Rw + Z)\left[(R - P)w + Z\right] + \frac{\Omega^2 c^2}{4A}P$$

is positive.

Substituting, we see that the motion will be stable for any value of Ω if

$$\left(w + \frac{\kappa c}{\pi a^2}\right)\left(w + \frac{\kappa c \rho}{2\pi a^2 (\rho + \rho')}\right) > 0.$$

Therefore the motion is stable for all positive values of w, and for all negative values numerically less than

$$\frac{\kappa c}{\pi a^2} \frac{\rho}{2(\rho + \rho')},$$

or numerically greater than

$$\frac{\kappa c}{\pi a^2}.$$

The motion is stable for values of w between these values if

$$\Omega^2 > -\frac{2\rho(\rho + \rho')^2}{\rho'^2(\rho + 2\rho')}\left(w + \frac{\kappa c}{\pi a^2}\right)\left(w + \frac{\kappa c}{\pi a^2}\frac{\rho}{2(\rho + \rho')}\right).$$

The greatest value the right-hand side can have is

$$\frac{\kappa^2 c^2}{\pi^2 a^4}\frac{\rho(3\rho + 2\rho')}{\rho'^2}.$$

Therefore the motion is always stable if

$$\Omega > \frac{\kappa c}{\pi a^2}\sqrt{\frac{\rho(3\rho + 2\rho')}{\rho'^2}}.$$

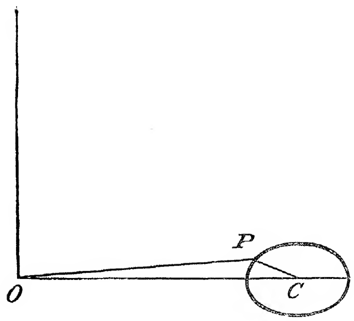
Another possible steady motion of the ring is for it to move round in a circle, as if it were a rigid body attached to an axis in the plane of the ring. Mr. BASSET shows that if T be the time of a complete oscillation, and r the radius of the circle described by the centre of the ring, then $r = \mathrm{T}/r\pi \,.\, \mathrm{Z}/\mathrm{R}$. When the ring is very fine this becomes

$$r = \frac{\mathrm{T}}{16\pi^2}\frac{c\kappa}{a^2}\frac{\rho}{\rho + \rho'}.$$

Section V.—*Annular Form of Rotating Fluid.*

§ 20. In order that a surface may be a possible figure of equilibrium of rotating fluid, it is necessary that $\mathrm{V} + \frac{1}{2}\omega^2 r^2 \sin^2\theta$ should be constant over the boundary.

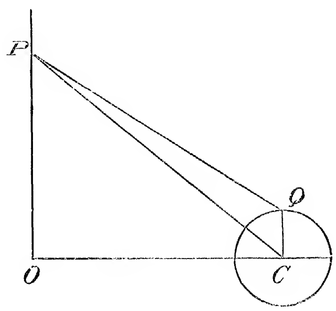
Let us assume that $\rho = a(1 + \beta_1 \cos\chi + \beta_2 \cos 2\chi + \beta_3 \cos 3\chi + \ldots)$ is the equation to the cross-section of the annulus, and that β_1, β_2, &c., are small quantities.



Then, by making C the centre of gravity of the cross-section, we obtain the equation $\beta_1 + \beta_1\beta_2 + \beta_2\beta_3 + \ldots = 0$; so that β_1 vanishes compared with β_2.

We shall show that β_2 is of order σ^2, β_3 of order σ^3, &c., where σ denotes a/c, and is taken fairly small.

To the first order of the small quantities β_2, β_3, β_4, the potential of the ring is the potential of the solid ring $r = a$, together with the potential of a distribution on the ring of surface density $a\ (\beta_2 \cos 2\chi + \beta_3 \cos 3\chi + \beta_4 \cos 4\chi)$, it will be most convenient to find the potential in this way, and take account of the terms arising from β_2^2 separately.



§ 21.—*To find the Potential of a Surface Distribution* $a\beta_2 \cos 2\chi$ *on the Ring at a Point on the Axis.*

Let

$$\angle\,\mathrm{OCQ} = \chi : \angle\,\mathrm{OCP} = \alpha : \angle\,\mathrm{PCQ} = \phi.$$
$$\mathrm{OC} = c : \mathrm{CQ} = a : \mathrm{CP} = \mathrm{R}.$$

Then

$$\mathrm{V} = 2\pi a^2 \int_0^{2\pi} \frac{(c - a\cos\chi)\,\beta_2 \cos 2\chi}{\sqrt{(\mathrm{R}^2 - 2a\mathrm{R}\cos\phi + a^2)}}\, d\phi$$
$$= \frac{2\pi a^2 c\,\beta_2}{\mathrm{R}} \int_0^{2\pi} \left\{\cos 2(\phi + \alpha) - \frac{a}{2c}\cos(\phi + \alpha) - \frac{a}{2c}\cos 3(\phi + \alpha)\right\}$$
$$\left\{1 + \frac{a}{\mathrm{R}}\mathrm{P}_1 + \frac{a}{\mathrm{R}^2}\mathrm{P}_2 + \text{\&c.}\right\} d\phi$$
$$= \frac{2\pi a^2 c\beta_2}{\mathrm{R}} \int_0^{2\pi} \left\{\cos 2\phi \cos 2\alpha - \frac{\sigma}{2}\cos\phi\cos\alpha - \frac{\sigma}{2}\cos 3\phi\cos 3\alpha\right\}$$
$$\left\{1 + \frac{a}{\mathrm{R}}\mathrm{P}_1 + \frac{a^2}{\mathrm{R}^2}\mathrm{P}_2 + \ldots\right\} d\phi,$$

since such integrals as

$$\int_0^{2\pi} \sin 2\phi \, P_n(\cos\phi) \, d\phi$$

vanish. Therefore

$$V = \frac{2\pi a^2 c}{R}\beta_2 \cos 2\alpha \int_0^{2\pi}\left\{\frac{a^2}{R^2}\cos 2\phi \, P_2 + \frac{a^4}{R^4}\cos 2\phi \, P_4 + \frac{a^6}{R^6}\cos 2\phi \, P_6 + \ldots\right\} d\phi$$

$$+ \frac{2\pi a^2 c}{R}\beta_2 \cos \alpha \int_0^{2\pi}\left\{-\frac{\sigma}{2}\frac{a}{R}\cos\phi \, P_1 - \frac{\sigma}{2}\frac{a^3}{R^3}\cos\phi \, P_1 - \ldots\right\} d\phi$$

$$+ \frac{2\pi a^2 c}{R}\beta_2 \cos 3\alpha \int_0^{2\pi}\left\{-\frac{\sigma}{2}\frac{a^3}{R^3}\cos 3\phi \, P_3 - \frac{\sigma}{2}\frac{a^5}{R^5}\cos 3\phi \, P_5\right\} d\phi$$

$$= \beta_2 \frac{2\pi^2 a^2 c}{R}\cos 2\alpha \left\{\tfrac{3}{4}\frac{a^2}{R^2} + \tfrac{5}{16}\frac{a^4}{R^4} + \tfrac{105}{512}\frac{a^6}{R^6} + \ldots\right\}$$

$$- \beta_2 \frac{2\pi^2 a^2 c}{R}\frac{\sigma}{2}\cos\alpha\left\{\frac{a}{R} + \tfrac{3}{8}\frac{a^3}{R^3} + \ldots\right\}$$

$$- \beta_2 \frac{2\pi^2 a^2 c}{R}\frac{\sigma}{2}\cos 3\alpha\left\{\tfrac{5}{8}\frac{a^3}{R^3} + \tfrac{35}{128}\frac{a^5}{R^5} + \ldots\right\},$$

the integrals being found at once by using the expansions of P_1, P_2, &c., in cosines of multiples of ϕ.

Substituting now $\frac{c}{R}$ for $\cos\alpha$, $\frac{2c^2}{R^2} - 1$ for $\cos 2\alpha$, $\frac{4c^3}{R^2} - \frac{3c}{R}$ for $\cos 3\alpha$: we find for the value of the potential

$$V = 2\pi a^2 c\beta_2 \left\{-\tfrac{5}{4}\frac{a^2}{R^3} + \left(\tfrac{3}{2}a^2c^2 + \tfrac{7}{16}a^4\right)\frac{1}{R^5} - \tfrac{5}{8}a^4c^2\frac{1}{R^7} - \tfrac{35}{256}\frac{a^6c^2}{R^9} + \text{\&c.}\right\}$$

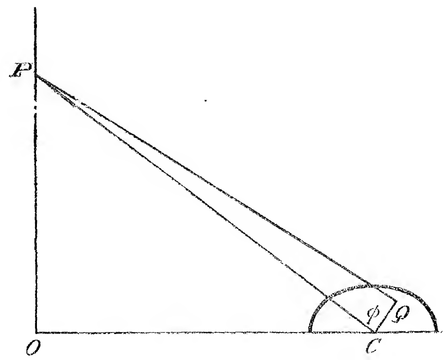
Similarly, the surface density $a\beta_3 \cos 3\chi$ gives

$$V = 2\pi a^2 c\beta_3 \left\{-\tfrac{21}{8}\frac{a^3 c}{R^5} + \tfrac{5}{2}\frac{a^3c^3}{R^7} - \tfrac{35}{32}\frac{a^5c^3}{R^9} + \ldots\right\},$$

and the surface density $a\beta_4 \cos 4\chi$ gives

$$V = 2\pi a^2 c\beta_4 \left\{\tfrac{35}{8}\frac{a^4c^4}{R^9}\right\}.$$

To find the part of the potential involving β_2^2.



The potential of the ring at P is given by

$$V = \int_0^{2\pi}\int_0^{\rho} 2\pi \frac{(c - r\cos\chi)\, r\, dr\, d\chi}{\sqrt{R^2 - 2Rr\cos\phi + r^2}}$$

$$= 2\pi \int_0^{2\pi} \left\{ \left(\frac{c\rho^2}{2} - \frac{\rho^3}{3}\cos\chi\right)\frac{1}{R} + \left(\frac{c\rho^3}{3} - \frac{\rho^4}{4}\cos\chi\right)\frac{\rho_1(\cos\phi)}{R^2} + \ldots \right\} d\chi.$$

Now

$$\rho = a(1 + \beta_2\cos 2\chi + \ldots),$$

therefore

$$\rho^2 = a^2\left\{1 + 2\beta_2\cos 2\chi + \ldots + \frac{\beta_2^2}{4} + \frac{\beta_2^2}{4}\cos 4\chi + \ldots\right\},$$

$$\rho^3 = a^3\left\{1 + 3\beta_2\cos 2\chi + \ldots + \frac{3\beta_2^2}{2} + \frac{3\beta_2^2}{2}\cos 4\chi + \ldots\right\},$$

&c.

Therefore, the terms which will give rise to terms of highest order in β_2^2 at surface of ring are

$$2\pi a^2 c\beta_2^2 \int_0^{2\pi} \left\{\frac{1}{4R} + \tfrac{5}{4}\frac{a^4}{R^5}\cos 4\chi\, P_4(\cos 4\phi)\right\} d\phi$$

$$= 2\pi^2 a^2 c\beta_2^2 \left\{\frac{1}{2R} + \tfrac{5}{4}\cdot\tfrac{35}{64}\cos 4\alpha\,\frac{a^4}{R^5}\right\}$$

$$= 2\pi^2 a^2 c\beta_2^2 \left\{\frac{1}{2R} + \tfrac{175}{32}\frac{a^4 c^4}{R^9}\right\}.$$

Collecting these results, the potential of the ring at P is given by

$$\frac{V}{2\pi a^2 c} = \frac{1}{R}\left(1 + \frac{\beta_2^2}{2}\right) + \frac{ac}{R^3}\left(-\frac{\sigma}{8} - \tfrac{5}{4}\beta_2\sigma\right) + \frac{a^2c^2}{R^5}\left\{-\frac{\sigma^2}{64} + \beta_2\left(\tfrac{3}{2} + \tfrac{7}{16}\sigma^2\right) - \tfrac{21}{8}\sigma\beta_3\right\}$$

$$+ \frac{a^3c^3}{R^7}\left\{-\frac{5\sigma^3}{1024} - \tfrac{5}{8}\sigma\beta_2 + \tfrac{5}{2}\beta_3\right\} + \frac{a^4c^4}{R^9}\left\{-\tfrac{35}{2^{14}}\sigma^4 - \tfrac{35}{256}\sigma^2\beta_2 - \tfrac{35}{32}\sigma\beta_3 + \tfrac{35}{8}\beta_4 + \tfrac{175}{32}\beta_2^2\right\}$$

$$+ \&c.$$

§ 22. The potential at any point external to the ring whose polar coordinates are r, θ, is found by writing

$$\frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\sqrt{(r^2+c^2-cr\sin\theta\cos\phi)}}, \text{ \&c.,}$$

instead of $1/R$, $1/R^3$, &c., as in Section II., § 5.

Instead of $1/R$, $1/R^3$, $1/R^5$, &c., we write

$$\frac{I_1}{\pi},\quad \frac{I_2}{\pi},\quad \frac{I_3}{3\pi},\quad \frac{I_4}{3.5\pi},\quad \frac{I_5}{3.5.7\pi}.$$

Therefore, the potential at any point outside the ring is given by

$$\frac{V}{2\pi a^2 c} = A_1 I_1 + A_2\, ac\, I_2 + A_3 a^2c^2\, I_3 + \text{\&c.},$$

where

$$A_1 = 1 + \frac{\beta_2{}^2}{2}$$

$$A_2 = -\frac{\sigma}{8} - \tfrac{5}{4}\sigma\beta_2$$

$$A_3 = -\frac{\sigma^2}{192} + \beta_2\left(\tfrac{1}{2} + \tfrac{7}{48}\sigma^2\right) - \tfrac{7}{8}\sigma\beta_3$$

$$A_4 = \tfrac{1}{3}\left(-\frac{\sigma^3}{1024} - \tfrac{1}{8}\sigma\beta_2 + \tfrac{1}{2}\beta_3\right)$$

$$A_5 = \tfrac{1}{3}\left(-\frac{\sigma^4}{2^{14}} - \frac{\sigma^2\beta_2}{256} - \frac{\sigma\beta_3}{32} + \frac{\beta_4}{8} + \frac{5\beta_2{}^2}{32}\right).$$

Using the expansions given in § 3 (A), the potential at a point R, χ near the surface of the ring is given by

$$\begin{aligned}\frac{V}{2\pi a^2} = {} & A_1\left(l + 2 + \frac{2l+1}{16}s^2 + \frac{108l+27}{2048}s^4\right) + A_2\sigma\left(\frac{2l+3}{4} + \frac{12l+5}{128}s^2\right) + A_3\sigma^2\frac{36l+51}{32} \\ & + \cos\chi\left\{A_1\left(\frac{l+1}{2}s + \frac{9l+3}{64}s^3\right) + A_2\sigma\left(\frac{1}{s} + \frac{12l+11}{32}s\right) + A_3\sigma^2\frac{9}{4s}\right\} \\ & + \cos 2\chi\left\{A_1\left(\frac{3l+2}{16}s^2 + \frac{20l+4\frac{1}{3}}{256}s^4\right) + A_2\sigma\left(\tfrac{1}{4} + \frac{12l+9}{64}s^2\right)\right. \\ & \qquad\qquad\left. + A_3\sigma^2\left(\frac{1}{s^2} + \tfrac{3}{4}\right) + A_4\sigma^3\frac{5}{s^2}\right\} \\ & + \cos 3\chi\left\{A_1\frac{5l+2\frac{1}{3}}{64}s^3 + A_2\sigma\tfrac{3}{32}s + A_3\sigma^2\frac{1}{4s} + A_4\sigma^3\frac{2}{s^3}\right\} \\ & + \cos 4\chi\left\{A_1\frac{35l+11\frac{1}{3}}{1208}s^4 + A_2\sigma\tfrac{5}{128}s^2 + A_3\sigma^2\tfrac{3}{32} + A_4\sigma^3\frac{1}{2s^2} + A_5\sigma^4\frac{6}{s^4}\right\}\end{aligned}$$

where as before, l stands for $\log(8c/R) - 2$, s for R/c, and σ for a/c.

§ 23. At the surface of the ring

$$V + \frac{\omega^2 \varpi^2}{2}$$

is constant. Therefore

$$V + \frac{\omega^2}{2}(c - R\cos\chi)^2$$

is constant at the surface of the ring, or

$$\frac{V}{2\pi a^2} + \frac{\omega^2 c^2}{4\pi a^2}(1 - s\cos\chi)^2$$

is constant, when

$$s = \sigma(1 + \beta_2 \cos 2\chi + \beta_3 \cos 3\chi + \beta_4 \cos 4\chi).$$

Let

$$\frac{V}{2\pi a^2} + \frac{\omega^2 c^2}{4\pi a^2}(1 - s\cos\chi)^2$$

be called $f(s)$.

Then

$$f(s) = f(\sigma) + f'(\sigma)\,\sigma(\beta_2 \cos 2\chi + \ldots) + \tfrac{1}{2} f''(\sigma)\,\sigma^2 \beta_2^2 \frac{1 + \cos 4\chi}{2}.$$

Now

$$\begin{aligned} f'(s) = {} & A_1\left(-\frac{1}{s} + \frac{l}{4}s\right) - A_2\sigma\frac{1}{2s} + \frac{\omega^2 c^2}{4\pi a^2}s \\ & + \cos\chi\left\{A_1\frac{l}{2} - \frac{A_2\sigma}{s^2} - \frac{\omega^2 c^2}{2\pi a^2}\right\}\cos\chi \\ & + \cos 2\chi\left\{A_1\frac{6l+1}{16}s - \frac{2A_3\sigma^2}{s^3} + \frac{\omega^2 c^2}{4\pi a^2}s\right\} \end{aligned}$$

and

$$f''(s) = \left(\frac{A_1}{s^2} + \frac{\omega^2 c^2}{4\pi a^2}\right) + \frac{\omega^2 c^2}{4\pi a^2}\cos 2\chi.$$

Therefore, $\sigma f'(\sigma)(\beta_2 \cos 2\chi + \&c.)$

$$\begin{aligned} = {} & \left(A_1\frac{6\lambda+1}{32}\sigma^2 - A_3 + \frac{\omega^2 c^2}{4\pi a^2}\cdot\frac{\sigma^2}{2}\right)\beta_2 \\ & + \left(A_1\frac{\lambda}{4}\sigma - \frac{A_2}{2} - \frac{\omega^2 c^2}{4\pi a^2}\sigma\right)\beta_2\cos\chi \\ & + \left\{\left[A_1\left(-1 + \frac{\lambda}{4}\sigma^2\right) - \frac{A_2}{2}\sigma + \frac{\omega^2 c^2}{4\pi a^2}\right]\beta_2 + A_1\left[\frac{\lambda}{4}\sigma - \frac{A_2}{2} - \frac{\omega^2 c^2}{4\pi a^2}\sigma\right]\beta_3\right\}\cos 2\chi \\ & + \left\{\left(A_1\frac{\lambda}{4}\sigma - \frac{A^2}{2} - \frac{\omega^2 c^2}{4\pi a^2}\sigma\right)\beta_2 - A_1\beta_3\right\}\cos 3\chi \\ & + \left\{\left(A_1\frac{6\lambda+1}{32}\sigma^2 - A_3 + \frac{\omega^2 c^2}{4\pi a^2}\frac{\sigma^2}{2}\right)\beta_2 + \left(A_1\frac{\lambda}{4}\sigma - \frac{A_2}{2} - \frac{\omega^2 c^2}{4\pi a^2}\sigma\right)\beta_3 - A_1\beta_4\right\}\cos 4\chi, \end{aligned}$$

and

$$\frac{\sigma^2}{2} f''(\sigma)\, \beta_2^2 \frac{1 + \cos 4\chi}{2} = \frac{A_1}{4} \beta_2^2 + \frac{A_1}{4} \beta_2^2 \cos 4\chi.$$

All terms which are of order higher than σ^4 have been neglected; and throughout, it has been assumed that $\omega^2 c^2/4\pi a^2$ is of zero order in σ, β_2 of the second order, β_3 of the third, and so on—an assumption which will be justified by the result.

The value of

$$\frac{V}{2\pi a^2} + \frac{\omega^2 c^2}{4\pi a^2} (1 - s \cos \chi)^2$$

at the surface of the ring is

$$\begin{aligned}
&A_1 \left(\lambda + 2 + \frac{2\lambda + 1}{16} \sigma^2 + \frac{108\lambda + 27}{2048} \sigma^4 + \frac{6\lambda + 1}{32} \sigma^2 \beta_2 + \tfrac{1}{4} \beta_2^2\right) \\
&\qquad + A_2 \left(\frac{2\lambda + 3}{4} \sigma + \frac{12\lambda + 5}{128} \sigma^3\right) + A_3 \left(\frac{36\lambda + 51}{32} \sigma^2 - \beta_2\right) + \frac{\omega^2 c^2}{4\pi a^2} \left(1 + \frac{\sigma^2}{2} + \beta_2 \frac{\sigma^2}{2}\right) \\
&+ \cos \chi \left\{ A_1 \left(\frac{\lambda + 1}{2} \sigma + \frac{9\lambda + 3}{64} \sigma^3 + \frac{\lambda}{4} \sigma \beta_2\right) + A_2 \left(1 + \frac{12\lambda + 11}{32} \sigma^2 - \frac{\beta_2}{2}\right) \right. \\
&\qquad\qquad \left. + A_3 \tfrac{9}{4} \sigma + \frac{\omega^2 c^2}{4\pi a^2} (-2\sigma - \sigma\beta_2) \right\} \\
&+ \cos 2\chi \left\{ A_1 \left(\frac{3\lambda + 2}{16} \sigma^2 + \frac{20\lambda + 4\frac{1}{3}}{256} \sigma^4 - \beta_2 + \frac{\lambda}{4} \sigma^2 \beta_2 + \frac{\lambda}{4} \sigma \beta_3\right) \right. \\
&\qquad + A_2 \left(\frac{\sigma}{4} + \frac{12\lambda + 9}{64} \sigma^3 - \frac{\sigma}{2} \beta_2 - \tfrac{1}{2} \beta_3\right) + A_3 (1 + \tfrac{3}{4} \sigma^2) \\
&\qquad\qquad \left. + A_4\, 5\sigma + \frac{\omega^2 c^2}{4\pi a^2} \left(\frac{\sigma^2}{2} + \beta_2 - \sigma\beta_3\right) \right\} \\
&+ \cos 3\chi \left\{ A_1 \left(\frac{5\lambda + 2\frac{1}{3}}{64} \sigma^3 + \frac{\lambda}{4} \sigma \beta_2 - \beta_3\right) + A_2 \left(\tfrac{3}{32} \sigma^2 - \tfrac{1}{2} \beta_2\right) \right. \\
&\qquad\qquad \left. + A_3 \frac{\sigma}{4} + 2A_4 - \frac{\omega^2 c^2}{4\pi a^2} \sigma \beta_2 \right\} \\
&+ \cos 4\chi \left\{ A_1 \left(\frac{35\lambda + 11\frac{1}{3}}{1024} \sigma^4 + \frac{6\lambda + 1}{32} \sigma^2 \beta_2 + \frac{\lambda}{3} \sigma \beta_3 - \beta_4 + \frac{\beta_2^2}{4}\right) \right. \\
&\qquad + A_2 \left(\tfrac{5}{128} \sigma^3 - \tfrac{1}{2} \beta_3\right) + A_3 \left(\tfrac{3}{32} \sigma^2 - \beta_2\right) \\
&\qquad\qquad \left. + A_4 \frac{\sigma}{2} + 6A_5 + \frac{\omega^2 c^2}{4\pi a^2} \left(\frac{\sigma^2}{2} \beta_2 - \sigma \beta_3\right) \right\}.
\end{aligned}$$

Equating the coefficients of $\cos \chi$, $\cos 2\chi$, $\cos 3\chi$, $\cos 4\chi$ to zero, we obtain four equations to find

$$\frac{\omega^2 c^2}{4\pi a^2}, \ \beta_2, \ \beta_3, \text{ and } \beta_4.$$

Substituting the values of A_1, A_2, &c., these equations are

$$\left.\begin{aligned}
&-\frac{\omega^2 c^2}{2\pi a^2}\left(1+\frac{\beta_2}{2}\right)+\frac{4\lambda+3}{8}+\frac{12\lambda-1}{128}\sigma^2+\frac{4\lambda-1}{16}\beta_2=0,\\
&\frac{3\lambda+\frac{17}{12}}{16}\sigma^2+\frac{7\lambda-\frac{19}{24}}{128}\sigma^4+\frac{\omega^2c^2}{\pi a^2}\frac{\sigma^2}{8}+\beta_2\left\{-\tfrac{1}{2}+\frac{4\lambda+1}{16}\sigma^2+\frac{\omega^2c^2}{\pi a^2}\frac{\sigma^2}{4}\right\}\\
&\qquad\qquad+\beta_3\left\{\left(\lambda-\tfrac{1}{12}\right)\frac{\sigma}{4}-\frac{\omega^2c^2}{\pi a^2}\frac{\sigma}{4}\right\}=0,\\
&\frac{\lambda+\frac{35}{24}}{64}\sigma^3+\left(\lambda+\tfrac{5}{12}-\frac{\omega^2c^2}{\pi a^2}\right)\frac{\sigma}{4}\beta_2-\tfrac{2}{3}\beta_3=0,\\
&\frac{35\left(\lambda+\frac{19}{120}\right)}{1024}\sigma^4+\frac{3\lambda+\frac{7}{8}}{16}\sigma^2\beta_2+\frac{\lambda+\frac{1}{3}}{4}\sigma\beta_3-\tfrac{3}{4}\beta_4+\frac{\beta_2^2}{16}+\frac{\omega^2c^2}{\pi a^2}\left(\frac{\sigma^2}{8}\beta_2-\frac{\sigma}{4}\beta_3\right)=0.
\end{aligned}\right\}$$

These equations give

$$\left.\begin{aligned}
\frac{\omega}{\pi}&=\left(\lambda+\tfrac{3}{4}\right)\sigma^2-\tfrac{1}{8}\left(\lambda+\tfrac{19}{12}\right)\sigma^4\\
\beta_2&=\frac{\frac{5}{8}\left(\lambda+\frac{7}{12}\right)\sigma^2+\frac{3}{64}\left(\lambda-\frac{11}{12}\right)\sigma^4}{1-\left(\lambda+\frac{1}{2}\right)\sigma^2}\\
\beta_3&=\tfrac{5}{128}\left(\lambda-\tfrac{7}{24}\right)\sigma^3\\
\beta_4&=\frac{75\lambda^2+80\lambda+21}{256}\sigma^4,
\end{aligned}\right\}$$

where $\sigma = a/c$ and $\lambda = \log_e(8c/a) - 2$.

The result $\omega^2/\pi = (\lambda + \frac{3}{4})\sigma^2$ is given by MME. KOWALEWSKI in a paper on Saturn's rings in the 'Astronomische Nachrichten' for 1885. She finds $\beta_2 = \frac{1}{2}(\lambda + \frac{3}{4})\sigma^2$. M. POINCARÉ also gives $\omega^2/\pi = (\lambda + \frac{3}{4})\sigma^2$, and finds $\beta_2 = \frac{3}{4}(\lambda + \frac{3}{4})\sigma_2$. Both papers are given in TISSERAND'S 'Mécanique Céleste,' vol. 2. The value found above has been kindly verified for me by Mr. HERMAN, Fellow of Trinity College, Cambridge.

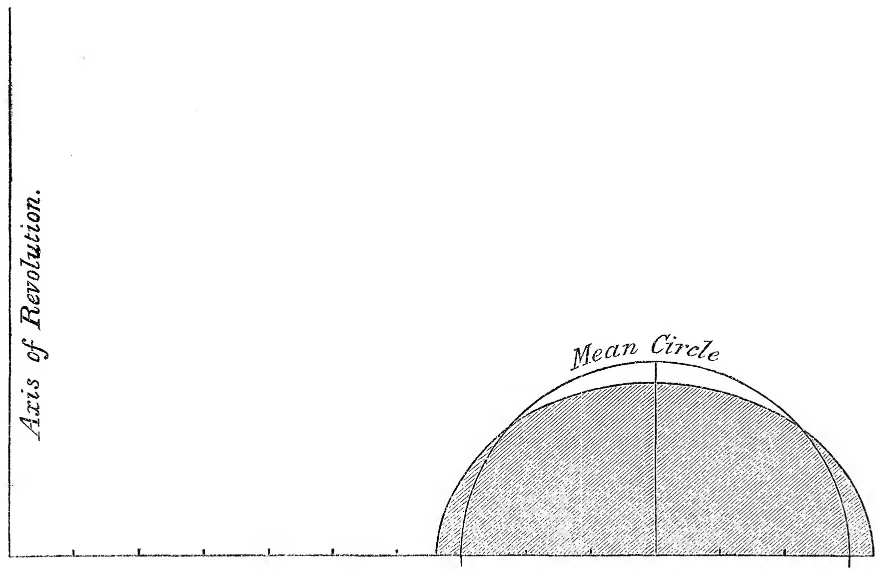
The numerical values of ω^2/π, β_2, &c., are given below when $\sigma = \frac{1}{10}, \frac{2}{10}, \frac{3}{10}$.

$\sigma = \cdot 1$	$\lambda = 2\cdot 3820$	$\beta_2 = \cdot 0189$	$\beta_3 = \cdot 0001$	$\beta_4 = \cdot 0003$	$\frac{\omega^2}{\pi\rho} = \cdot 0313$
$\sigma = \cdot 2$	$\lambda = 1\cdot 6889$	$\beta_2 = \cdot 0570$	$\beta_3 = \cdot 0004$	$\beta_4 = \cdot 0023$	$\frac{\omega^2}{\pi\rho} = \cdot 0970$
$\sigma = \cdot 3$	$\lambda = 1\cdot 2834$	$\beta_2 = \cdot 1268$	$\beta_3 = \cdot 0010$	$\beta_4 = \cdot 0078$	$\frac{\omega^2}{\pi\rho} = \cdot 1836.$

A figure is given for the case of $\sigma = \cdot 3$.

The cross-section is roughly an ellipse whose major axis is perpendicular to the axis of the ring. The eccentricity of this ellipse increases with the angular velocity.

[When the paper was read, numerical results were given for larger values of σ. The above method cannot, however, be employed in such cases. For, as the



eccentricity of the cross-section increases, the expression given for the potential at external points will become divergent at the extremity of the minor axis of the cross-section. A similar result will occur if the potential of an elliptic cylinder be found at external points, considering it as an approximation to a circular cylinder. In this case it is easy to show that the eccentricity must be $< 1/\sqrt{2}$. Assuming the same result for the case of a ring, it will be seen that $\beta_2 = \cdot 2076$, the value the above method gives for $\sigma = \cdot 4$, gives the cross-section of too great eccentricity for the series to be convergent. That the series were divergent for large values of β_2 was also suggested by one of the Referees. Aug., 1892.]